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Triangular Poisson Structures on Lie Groups and Symplectic Reduction

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Abstract. We show that each triangular Poisson Lie group can be decomposed into Poisson submanifolds each of which is a quotient of a symplectic manifold. The Marsden–Weinstein–Meyer symplectic reduction technique is then used to give a complete description of the symplectic foliation of all triangular Poisson structures on Lie groups. The results are illustrated in detail for the generalized Jordanian Poisson structures on $SL(n)$.

1. Introduction

A Poisson group structure on a Lie group G is given by specifying a Lie bialgebra structure on its Lie algebra $\mathfrak{g}$. Such a Lie bialgebra structure is said to be triangular if the cocommutator is induced from a skew-symmetric solution $r \in \mathfrak{g} \wedge \mathfrak{g}$ of the classical Yang–Baxter equation. A triangular Poisson Lie group is a Lie group equipped with such a Poisson group structure, associated to a triangular Lie bialgebra structure on $\mathfrak{g}$. If we denote the left and right invariant vector fields on G, induced by an element $x \in \mathfrak{g} \cong T_e G$ by $\mathcal{L}_x$ and $\mathcal{R}_x$ and if $r = \sum r_i^1 \wedge r_i^2$, then the Poisson bivector field on G is given by

$$\pi_r = \sum \mathcal{L}_{r_i^1} \wedge \mathcal{L}_{r_i^2} - \sum R_{r_i^1} \wedge \mathcal{R}_{r_i^2}. \tag{1.1}$$

In this article we give a description of the symplectic leaves of all triangular Poisson Lie groups. From the point of view of quantization, our results are important because the primitive ideals of the quantization are generally closely connected with the closures of the symplectic leaves. Since the primitive ideals are the annihilators of the irreducible representations of an algebra this yields an important link between representations of the quantization and symplectic leaves of the Poisson manifold. The most well known examples of this phenomenon are the Kostant–Kirillov structure on the dual $\mathfrak{g}^*$ of a solvable algebraic Lie algebra and the standard Poisson structure on a semisimple Lie group. In both cases there

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is a natural bijection between the primitive ideals of the quantization and the symplectic leaves of the Poisson Lie group. In the former case, the quantization is the enveloping algebra and the symplectic leaves are precisely the coadjoint orbits [**D**]. In the latter case the quantization is the Drinfeld–Jimbo quantum group which played a major role in many recent developments in mathematical physics and pure mathematics [**CP**]. The symplectic leaves in this case are closely related to double Bruhat cells in semisimple Lie groups [**J, HL, HKKR, KZ, Y**]. A second important stimulus for studying symplectic leaves in Poisson Lie groups is their relation to integrable systems. Poisson Lie groups provide a very general framework for constructing integrable systems [**STS3, R, HKKR**] whose phase spaces are the leaves of those Poisson structures. Detailed geometric information for the latter is crucial in understanding the properties of such dynamical systems.

The main tool for studying the leaves of a Poisson Lie group G is Semenov-Tian-Shansky's (local) dressing action of the dual Poisson Lie group G^* on G. The problem of explicit parametrization of the orbits of this actions is highly nontrivial and was only solved for specific classes of Poisson Lie groups see [**LS, KS, LW, HL**], most generally for any Belavin–Drinfeld structure on a complex simple Lie group [**Y**]. The problem of understanding the geometry of leaves is open even in the simplest case of the standard Poisson structure on a complex simple Lie group. In this paper we show that one can study each triangular Poisson structure on a Lie group and in particular its symplectic leaves using different methods, namely Marsden–Weinstein–Meyer symplectic reduction. Our methods provide an explicit description of the symplectic leaves of any triangular Poisson Lie group and provide a framework for the future study of the geometry of these leaves. The approach applies to both complex and real groups. All manifolds in this paper can be of either type. For shortness we denote

$$\mathrm{k} = \mathbb{C} \quad \text{or} \quad \mathbb{R}, \tag{1.2}$$

according to whether the manifold in discussion is complex or real.

Suppose that $G \times M \to M$ is a free, proper, symplectic action of a connected Lie group G on a connected symplectic manifold $(M, \pi^{-1} \in \wedge^2 T^*M)$ for a non-degenerate Poisson structure $\pi \in \wedge^2 TM$. Any smooth function f on M defines the Hamiltonian vector field $X_f = df \rfloor \pi$. The Lie algebra $\mathfrak{g}$ of G acts naturally as vector fields on M. Denote the vector field on M induced by an element $x \in \mathfrak{g}$ by x_M. Recall that a moment map for this action is a map $J \colon M \to \mathfrak{g}^*$ such that $X_{J^*(x)} = x_M$. If such a map exists, the action is called *weakly Hamiltonian.* In general a moment map will not be Poisson or, equivalently, G equivariant. It is made such by several twistings (equivalently by passing to central extensions):

1.) The deviation from G equivariance is measured by a one cocycle $\mathcal{B} \colon G \to \mathfrak{g}^*$ defined by $\mathcal{B}(g) = J(gm) - \mathrm{Ad}^*_g(J(m))$ which is independent of $m \in M$. The coadjoint action of G on $\mathfrak{g}^*$ can then be twisted to a new geometric action by defining $g * \xi = \mathrm{Ad}^*_g(\xi) + \mathcal{B}(g)$. The moment map $J \colon M \to \mathfrak{g}^*$ is G equivariant with respect to this new action of G on $\mathfrak{g}^*$.

2.) The derivative $\gamma = d_e\mathcal{B}$, which is a Lie algebra 1-cocycle $\gamma \colon \mathfrak{g} \to \mathfrak{g}^*$, gives rise to the 2-cocycle for $\mathfrak{g}$

$$\beta \colon \mathfrak{g} \wedge \mathfrak{g} \to \mathrm{k}, \quad \beta(x, y) = -\langle \gamma(x), y \rangle = \langle x, \gamma(y) \rangle, \tag{1.3}$$

cf. (1.2) ($e \in G$ is the identity element). It can be equivalently defined by

$$\beta(x, y) = \{J^*(x), J^*(y)\}$$

which is necessarily a constant function on M. The cocycle β induces the affine Poisson structure on $\mathfrak{g}^*$, defined by

$$\{x, y\} = [x, y] + \beta(x, y),\ \forall x, y \in \mathfrak{g} \subset \mathrm{k}[\mathfrak{g}^*]. \tag{1.4}$$

Alternatively one arrives at it by considering a central extension of $\mathfrak{g}$ by the 2-cocycle (1.3) and then restricting the Kirillov–Kostant Poisson structure to a hyperplane in its dual, identified with $\mathfrak{g}^*$. For this new Poisson structure on $\mathfrak{g}^*$ the moment map $J\colon M \to \mathfrak{g}^*$ is Poisson. The symplectic leaves of (1.4) are the orbits of the $\mathcal{B}$-twisted coadjoint action of G on $\mathfrak{g}^*$.

Finally the orbit space M/G has a natural manifold structure and inherits the Poisson structure $p_*(\pi)$ from M. The Marsden–Weinstein–Meyer [**MW, M**] reduction theorem states that the symplectic leaves of $(M/G, p_*(\pi))$ can be described as the connected components of $p(J^{-1}(\mathcal{O}))$ where $\mathcal{O}$ is an orbit for the $\mathcal{B}$-twisted coadjoint action of G on $\mathfrak{g}^*$ and p is the natural projection $p\colon M \to M/G$. The symplectic manifolds $(p(J^{-1}(\mathcal{O})), p_*(\pi)^{-1})$ are called symplectic reductions of (M, π^{-1}) with respect to G, $\mathcal{O}$. The connectedness of G and M is essential in establishing the properties of the moment map J for the weak Hamiltonian action, and the properties of the cocycles $\mathcal{B}$, γ and β. On the other hand if these properties are satisfied, then the symplectic reduction technique is applicable even if both G and M are disconnected. For more details on weakly Hamiltonian actions and moment maps we refer to the books [**AM, CdSW**].

Let (G, π_r) be a triangular Poisson Lie group. The subspace $\mathfrak{p} = \{(\xi \otimes \mathrm{Id})(r) \mid \xi \in \mathfrak{g}^*\}$ (i.e. the span of all components of the r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$) is a Lie subalgebra of $\mathfrak{g}$ which has a canonical structure of a quasi-Frobenius Lie algebra, see Section 2 for details. Moreover r is a non-degenerate skew-symmetric solution of the classical Yang–Baxter equation on $\mathfrak{p}$, in particular the map $\check{r}\colon \mathfrak{p}^* \to \mathfrak{p}$ is a linear isomorphism. The inverse map $\check{r}^{-1}\colon \mathfrak{p} \to \mathfrak{p}^*$ is a Lie algebra 1-cocycle. Denote by P the connected subgroup of G corresponding to $\mathfrak{p}$. (In general P is not a closed subgroup of G.) Let $\eta\colon \widetilde{P} \to P$ be a connected cover for which there exists a group cocycle $\mathcal{B}\colon \widetilde{P} \to \mathfrak{p}^*$ such that $d_e\mathcal{B} = \check{r}^{-1}$. In Lemma 3.3 we show that all double cosets PtP are complete Poisson submanifolds of G. By a complete Poisson submanifold of a given Poisson manifold we mean a submanifold which is stable under all Hamiltonian flows, i.e. a submanifold which is a union of symplectic leaves, see Section 3 for details. Thus the symplectic foliation of (G, π_r) is the union of the symplectic foliations of these double cosets. Each of them can be obtained as a quotient of a a symplectic manifold as follows. Fix $t \in G$. The group $\widetilde{P} \times \widetilde{P}$ has a natural left invariant symplectic structure $(-\mathcal{L}_r, \mathcal{L}_r)$, see (2.1) and Lemma 2.1 below. In Section 3 we show that the map

$$\nu_t\colon \widetilde{P} \times \widetilde{P} \to PtP, \quad \nu_t(p_1, p_2) = \eta(p_1^{-1})t\eta(p_2), p_i \in \widetilde{P}$$

is Poisson and that the fibers of this map are exactly the left orbits of the subgroup

$$\widetilde{P}_t = \{(p_1, p_2) \in \widetilde{P} \times \widetilde{P} \mid \eta(p_2) = \mathrm{Ad}_t^{-1}\eta_1(p_1)\}$$

of $\widetilde{P} \times \widetilde{P}$. Its Lie algebra can be identified with $\mathfrak{p}_t = \mathfrak{p} \cap \mathrm{Ad}_t(\mathfrak{p})$. We further show that the action of $\widetilde{P}_t$ on $\widetilde{P} \times \widetilde{P}$ is weakly Hamiltonian with a moment map

$$J_t(p_1, p_2) = -\mathcal{B}(p_1) + \mathrm{Ad}^*_{t^{-1}}\mathcal{B}(p_2) + \mathfrak{p}_t^\perp \in \mathfrak{g}^*/(\mathfrak{p}_t^\perp) \cong \mathfrak{p}_t^*, \tag{1.5}$$

where for a subspace $V \subset \mathfrak{g}$ by $V^\perp$ we denote its orthogonal complement in $\mathfrak{g}^*$. Note that $\mathcal{B}$ takes values in $\mathfrak{p}^* \cong \mathfrak{g}^*/\mathfrak{p}^\perp$ and (1.5) makes sense because $\mathfrak{p}^\perp$ and

$\mathrm{Ad}_t(\mathfrak{p}^\perp)$ are contained in $\mathfrak{p}_t^\perp$. We refer to (3.8) for another way of writing the formula for the moment map $J_t(.,.)$. The associated group cocycle for this action is $\mathcal{B}_t = J_t|_{\widetilde{P}_t}$. Applying the reduction theory described above yields the description of the symplectic leaves of G.

THEOREM 1.1. *Let π_r be a triangular Poisson structure on a Lie group G. In the above notation the symplectic foliation of (G, π_r) is the union of the symplectic foliations of all double cosets PtP. For any $t \in G$ the symplectic leaves of $(PtP, \pi_r|_{PtP})$ are the connected components of $\nu_t(J_t^{-1}(\mathcal{O}))$ where $\mathcal{O}$ is an orbit for the $\mathcal{B}_t$-twisted coadjoint action of $\widetilde{P}_t$ on $\mathfrak{p}_t^*$.*

Let us note that the symplectic foliation of any Poisson group structure on a complex simple Lie group can be described on the basis of Theorem 1.1 and the results in [**Y**], which treated all Belavin–Drinfeld Poisson structures (i.e. all Poisson structures that are not triangular). It would be very interesting to find a noncommutative version of our symplectic reduction approach to study the primitive spectra of Hopf algebras that are quantizations of triangular Poisson structures on algebraic groups.

The proof of Theorem 1.1 is given in Section 3. The theorem simplifies in the important special case when $r \in \mathfrak{p} \wedge \mathfrak{p}$ is induced by a Frobenius structure on $\mathfrak{p}$. This is formulated in Proposition 3.5. Section 2 contains auxiliary results on symplectic structures related to triangular Poisson Lie groups and their symplectic reductions, as well as some background on Frobenius and quasi-Frobenius structures on Lie algebras. Section 4 treats generalized Jordanian Poisson structures on $SL(n)$ in detail.

The results in this paper were obtained in the Fall of 2000 and were reported at the conference on Quasiclassical and Quantum Structures at the Fields Institute (January 2001), and at seminars in Berkeley (November 2000) and Cornell (January 2002). We are grateful to Alan Weinstein whose comments at one of these talks helped us to improve our arguments. We are also thankful to him for his advice on notation, in particular he suggested to us the term complete Poisson submanifold.

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2. Quasi-Frobenius Lie algebras, symplectic forms and reduction

LEMMA 2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $r = \sum r_i^1 \wedge r_i^2 \in \mathfrak{g} \wedge \mathfrak{g}$. The following left G invariant bivector field*

$$\mathcal{L}_r = \sum \mathcal{L}_{r_i^1} \wedge \mathcal{L}_{r_i^2} \tag{2.1}$$

on G is Poisson if and only if r is a solution of the classical Yang–Baxter equation. In this case

$$\mathcal{R}_r = \sum \mathcal{R}_{r_i^1} \wedge \mathcal{R}_{r_i^2}$$

is a right invariant Poisson structure on G.

A proof of this Lemma can be found in [**CP**, Theorem 2.2.2].

Fix a Lie group G and a nondegenerate solution of the classical Yang–Baxter equation $r \in \mathfrak{g} \wedge \mathfrak{g}$ where $\mathfrak{g} = \mathrm{Lie}\, G$. The nondegeneracy condition means that $r \in \mathfrak{g} \wedge \mathfrak{g}$ defines a nondegenerate skew-symmetric (i.e. symplectic) form on $\mathfrak{g}^*$. The following relation to quasi-Frobenius Lie algebras is due to Stolin [**S**], see for details [**CP**, Section 3.1.D] and [**ES**, Section 3.5]. First the map

$$\check{r}\colon \mathfrak{g}^* \to \mathfrak{g}, \quad \text{given by} \quad \check{r}(\xi) = (\xi \otimes \mathrm{Id})r,\ \xi \in \mathfrak{g}^* \tag{2.2}$$

is an isomorphism between the underlying Lie algebra $\mathfrak{g}^*$ of the dual Lie bialgebra of $\mathfrak{g}$ and $\mathfrak{g}$.

1. The inverse $\check{r}^{-1}\colon \mathfrak{g} \to \mathfrak{g}^*$ is a Lie algebra 1-cocycle for the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

2. The bilinear form

$$\beta\colon \mathfrak{g} \wedge \mathfrak{g} \to \mathrm{k}, \quad \beta(x,y) = -\langle \check{r}^{-1}(x), y\rangle = \langle x, \check{r}^{-1}(y)\rangle \tag{2.3}$$

is a nondegenerate 2-cocycle for $\mathfrak{g}$ with values in the trivial representation, recall (1.2).

The second property is the definition of a quasi-Frobenius Lie algebra $(\mathfrak{g},\beta)$ and the above construction can be inverted to show that the class of those coincides with the class of triangular Lie bialgebras $(\mathfrak{g}, r)$ with a nondegenerate r-matrix. Lemma 2.1 implies that a Lie group G possesses a left invariant symplectic structure if and only if $(\mathrm{Lie}\,(G),\beta)$ is a quasi-Frobenius Lie algebra for some 2-cocycle β. Formulas (2.1) and (2.3) give the exact relation between the symplectic form and the 2-cocycle.

Recall also that a Frobenius Lie algebra is a pair $(\mathfrak{g},\xi)$ where $\mathfrak{g}$ is a Lie algebra and $\xi \in \mathfrak{g}^*$ is such that $x \otimes y \to \langle \xi, [x,y]\rangle$ is a nondegenerate form on $\mathfrak{g}$. The class of such is the subset of all quasi-Frobenius Lie algebras for which β is the coboundary of the 1-cochain $x \mapsto \langle \xi, x\rangle$, or equivalently $\check{r}^{-1}$ is the coboundary of the 0-cochain $\xi \in \mathfrak{g}^*$.

The following proposition describes under what conditions in the above situation the left action of G on $(G, \mathcal{L}_r)$ is weakly Hamiltonian.

Proposition 2.2. *Let G be a Lie group for which there exists a nondegenerate skew-symmetric solution $r \in \mathfrak{g}\wedge\mathfrak{g}$ of the classical Yang–Baxter equation. Let $(G, \mathcal{L}_r)$ be the corresponding left invariant symplectic structure on G. Then the following are equivalent.*

(1) *The left action of G is weakly Hamiltonian.*
(2) *There exists a 1-cocycle $\mathcal{B}\colon G \to \mathfrak{g}^*$ such that $d\mathcal{B}_e = \check{r}^{-1}$.*

In particular these conditions will hold if G is simply connected or if the 1-cocycle $\check{r}^{-1}\colon \mathfrak{g} \to \mathfrak{g}^*$ is a coboundary.

Proof. If $J\colon G \to \mathfrak{g}^*$ is a moment map for the left action of G on G, then $J - J(e)$ is another moment map which is necessarily a 1-cocycle. This follows from the fact that for any moment map $J\colon M \to \mathfrak{g}^*$ the function $\mathcal{B}(g) = J(gm) - \mathrm{Ad}^*_g(J(m))$ does not depend on $m \in M$ and is a 1-cocycle for G. Therefore, to prove the proposition, it is enough to observe that a 1-cocycle $\mathcal{B}\colon G \to \mathfrak{g}^*$ is a moment map if and only if $d_e\mathcal{B} = \check{r}^{-1}$ (where e denotes the identity element of G).

For any $x \in \mathfrak{g}$ and $f \in \mathcal{C}^\infty(G)$

$$\begin{aligned}
(X_{\mathcal{B}^*(x)} \cdot f)(e) &= \{\mathcal{B}^*(x), f(x)\}(e) \\
&= \sum \langle r_i^1, d_e(\mathcal{B}^*(x))\rangle \langle r_i^2, d_e f\rangle \\
&= \sum \langle \langle r_i^1, d_e\mathcal{B}(x)\rangle r_i^2, d_e f\rangle \\
&= \langle \check{r}(d_e\mathcal{B}(x)), d_e f\rangle.
\end{aligned}$$

On the other hand $x_G = \mathcal{R}_x$ and $(\mathcal{R}_x \cdot f)(e) = \langle x, d_e f\rangle$. Hence $X_{\mathcal{B}^*(x),e} = x_{G,e}$ if and only if $d_e\mathcal{B}(x) = \check{r}^{-1}(x)$. This proves the necessity of the condition.

Now assume that $\mathcal{B}$ is a cocycle and that $d_e\mathcal{B} = \check{r}^{-1}$. For $g \in G$ denote

$$\ell_g \colon \mathcal{C}^\infty(G) \to \mathcal{C}^\infty(G), \quad (\ell_g f)(g_1) = f(g_1 g^{-1}).$$

Notice that the cocycle condition implies that

$$\ell_g \mathcal{B}^*(x) = \mathcal{B}^*(\mathrm{Ad}_g x) - \mathcal{B}(g)(x)$$

and so $X_{\mathcal{B}^*(\mathrm{Ad}_g x)} = X_{\ell_g \mathcal{B}^*(x)}$. Thus

$$\begin{aligned}(X_{\mathcal{B}^*(x)} \cdot f)(g) &= (X_{\ell_{g^{-1}}\mathcal{B}^*(x)} \cdot \ell_{g^{-1}} f)(e) = (X_{\mathcal{B}^*(\mathrm{Ad}_{g^{-1}} x)} \cdot \ell_{g^{-1}} f)(e)\\ &= (\mathcal{R}_{\mathrm{Ad}_{g^{-1}} x} \cdot \ell_{g^{-1}} f)(e) = (\ell_{g^{-1}} \mathcal{R}_x \cdot f)(e)\\ &= (\mathcal{R}_x \cdot f)(g)\end{aligned}$$

Hence $X_{\mathcal{B}^*(x)} = \mathcal{R}_x = x_G$ and $\mathcal{B}$ is a moment map. □

Suppose that the left action of G on $(G, \mathcal{L}_r)$ is weakly Hamiltonian. Let K be a closed subgroup of G. The next Proposition describes the leaves of the reduction of $(G, \mathcal{L}_r)$ by K in the setting of Proposition 2.2. Let $\mathfrak{k} = \mathrm{Lie}\, K$ and $\mathrm{pr}_{\mathfrak{k}^*} : \mathfrak{g}^* \to \mathfrak{k}^*$ denote the projection associated to the inclusion $\mathfrak{k} \subset \mathfrak{g}$. If $\mathcal{B}$ is a moment map for the action of G on itself which is a group cocycle, then $J_K = \mathrm{pr}_{\mathfrak{k}^*} \circ \mathcal{B} \colon G \to \mathfrak{k}^*$ is a moment map for the left action of K on G. The cocycle for K associated to this weakly Hamiltonian map is $\mathcal{B}_K = \mathrm{pr}_{\mathfrak{k}^*} \circ \mathcal{B}|_K$ (because $\mathcal{B}(e) = 0$). Let $\nu \colon G \to K\backslash G$ be the natural projection. It induces the Poisson structure $\nu_*(\mathcal{L}_r)$ on $K\backslash G$ for which the symplectic reduction technique yields the following description of its symplectic leaves.

Proposition 2.3. *Let G be a Lie group for which $(\mathfrak{g} = \mathrm{Lie}\, G, \beta)$ is a Frobenius Lie algebra and let $r \in \mathfrak{g} \wedge \mathfrak{g}$ be the corresponding nondegenerate r-matrix. Assume that there exists a group 1-cocycle $\mathcal{B} \colon G \to \mathfrak{g}^*$ with $d_e\mathcal{B} = \check{r}^{-1}$. If K is a closed subgroup of G then the symplectic leaves of $(K\backslash G, \nu_*(\mathcal{L}_r))$ are the connected components of $\nu(J_K^{-1}(\mathcal{O}))$ where $\mathcal{O}$ is an orbit for the $\mathcal{B}_K$-twisted coadjoint action of K on $\mathfrak{k}^*$.*

Recall that the $\mathcal{B}_K$-twisted coadjoint action of K on $\mathfrak{k}^*$ is the action $k * \xi = \mathrm{Ad}_k^*(\xi) + \mathcal{B}_K(k)$.

Proposition 2.3 simplifies in the case of Frobenius Lie algebras.

Proposition 2.4. *Let G be a Lie group for which $(\mathfrak{g} = \mathrm{Lie}\, G, \xi \in \mathfrak{g}^*)$ is a quasi-Frobenius Lie algebra and let $r \in \mathfrak{g} \wedge \mathfrak{g}$ be the corresponding nondegenerate r-matrix. Then the symplectic leaves of $(K\backslash G, \nu_*(\mathcal{L}_r))$ are the connected components of $\nu(I_K^{-1}(\mathcal{O}))$ where*

$$I_K \colon G \to \mathfrak{k}^* \ \text{ is given by } \ I_K(g) = \mathrm{pr}_{\mathfrak{k}^*} \mathrm{Ad}_g^*(\xi), g \in G$$

and $\mathcal{O}$ is a coadjoint orbit of K.

Equivalently, the symplectic leaf of $(K\backslash G, \nu_*(\mathcal{L}_r))$ through Kg is the connected component of the submanifold

$$X_g = \{Kg' \in K\backslash G \mid g' \in G,\ \mathrm{Ad}_{g'}^*(\xi) \in \mathrm{Ad}_g^*(\xi) + \mathfrak{k}^\perp\} \subset K\backslash G,$$

containing Kg.

It is clear that only the coadjoint orbits in $\mathrm{pr}_{\mathfrak{k}^*}(\mathrm{Ad}_G^*(\xi))$ have nontrivial preimages and thus lead to symplectic leaves.

PROOF. In this case $\check{r}^{-1}\colon \mathfrak{g} \to \mathfrak{g}^*$ is coboundary: $\check{r}^{-1}(x) = \mathrm{ad}^*_x(\xi)$ for all $x \in \mathfrak{g}$. It integrates to the group coboundary $\mathcal{B}\colon G \to \mathfrak{g}^*$, $\mathcal{B}(g) = \mathrm{Ad}^*_g(\xi) - \xi$. Therefore the moment map J_K is

$$J_K(g) = \mathrm{pr}_{\mathfrak{k}^*}\mathcal{B}(g) = \mathrm{pr}_{\mathfrak{k}^*}\mathrm{Ad}^*_g(\xi) - \mathrm{pr}_{\mathfrak{k}^*}(\xi) = I_K(g) - \mathrm{pr}_{\mathfrak{k}^*}(\xi).$$

The corresponding group cocycle for K is the coboundary of $\mathrm{pr}_{\mathfrak{k}^*}(\xi)$ and the related twisted coadjoint orbits are just the translation of the coadjoint orbits of K by $-\mathrm{pr}_{\mathfrak{k}^*}(\xi)$. This implies the first part of the proposition.

For the second part one first observes that the symplectic leaf of $(K\backslash G, \nu_*(\mathcal{L}_r))$ through Kg is the connected component of

$$\{Kg' \in K\backslash G \mid g' \in G,\ \mathrm{Ad}^*_{g'}(\xi) \in \mathrm{Ad}^*_K \mathrm{Ad}^*_g(\xi) + \mathfrak{k}^\perp\},$$

containing Kg. This is exactly the set X_g because $\mathfrak{k}^\perp$ is Ad^*_K stable. □

3. Symplectic leaves of triangular Poisson Lie groups

Throughout this section we fix a triangular Poisson structure π_r on a Lie group G, cf. (1.1), where $r \in \mathfrak{g} \wedge \mathfrak{g}$ is a skew-symmetric solution of the classical Yang–Baxter equation on $\mathfrak{g} = \mathrm{Lie}\, G$. The subspace

$$\mathfrak{p} = \mathrm{Span}\{(\xi \otimes \mathrm{Id})r \mid \xi \in \mathfrak{g}^*\} \tag{3.1}$$

is a Lie subalgebra of $\mathfrak{g}$ because it is the image of the Lie algebra homomorphism $\check{r}\colon \mathfrak{g}^* \to \mathfrak{g}$, $\check{r}(\xi) = (\xi \otimes \mathrm{Id})r$ from the underlying Lie algebra of the dual Lie bialgebra $\mathfrak{g}^*$ of $\mathfrak{g}$ to $\mathfrak{g}$, see e.g. [**ES**]. It is clear that $r \in \mathfrak{p} \wedge \mathfrak{p}$ is a non-degenerate solution of the classical Yang–Baxter equation on $\mathfrak{p}$ which turns $(\mathfrak{p}, \beta)$ into a quasi-Frobenius Lie algebra for $\beta\colon \mathfrak{p} \wedge \mathfrak{p} \to \mathrm{k}$ given by (2.3), recall (1.2). Denote the connected subgroup of G with tangent Lie algebra $\mathfrak{p}$ by P. Note that in general P is not a closed subgroup of G. Recall from Section 2 that $\check{r}\colon \mathfrak{p}^* \to \mathfrak{p}$ is a linear isomorphism and its inverse $\check{r}^{-1}\colon \mathfrak{p} \to \mathfrak{p}^*$ is a 1-cocycle. It integrates to a group cocycle

$$\mathcal{B}\colon \widetilde{P} \to \mathfrak{p}^*,\ d_e\mathcal{B} = \check{r}^{-1} \tag{3.2}$$

for an appropriate (connected) cover $\widetilde{P}$ of P. Denote the projection $\eta\colon \widetilde{P} \to P$. We can always choose $\widetilde{P}$ to be the simply connected group with Lie algebra $\mathfrak{p}$. If $\check{r}^{-1}$ is a coboundary of $\xi \in \mathfrak{p}^*$ we can take $\widetilde{P} = P$ and set $\mathcal{B}(p) = \mathrm{Ad}^*_p(\xi) - \xi$, $\forall p \in P$.

First we define the notion of a complete Poisson submanifold.

DEFINITION 3.1. Let (M, π) be a Poisson manifold. A (not necessarily closed) submanifold X of M is called a complete Poisson submanifold if X is stable under all Hamiltonian flows on M.

If X is a complete Poisson submanifold of (M, π) then

$$\pi_x \in \wedge^2 T_x X\ \forall x \in X, \tag{3.3}$$

i.e. X is a Poisson submanifold of (M, π) in the local sense, see e.g. [**V**]. Conversely a Poisson submanifold X is not necessarily a complete Poisson submanifold of (M, π), this is only true if X is a closed submanifold of M. A submanifold X of (M, π) is a complete Poisson submanifold if and only if it is a union of symplectic leaves of M. The following Lemma clarifies the passage from Poisson submanifolds to complete Poisson submanifolds.

Lemma 3.2. *Assume that (M, π) is a Poisson manifold which can be decomposed into the disjoint union of (not necessarily closed) submanifolds $M = \sqcup_{\alpha \in A} X_\alpha$ satisfying*

$$\pi_x \in \wedge^2 T_x X_\alpha \ \forall x \in X_\alpha, \tag{3.4}$$

i.e. all X_α are Poisson submanifolds of (M, π). Then all X_α are complete Poisson submanifolds of (M, π).

Proof. If $\gamma\colon (a, b) \to M$ is a Hamiltonian path then (3.4) implies that $\gamma^{-1}(X_\alpha)$ is an open subset of (a, b) for all $\alpha \in A$. Since (a, b) is connected the image of γ sits within a single submanifold X_α of (M, π) which is easily seen to complete the proof of the Lemma. □

Lemma 3.3. *Assume that (G, π_r) is a triangular Poisson Lie group and P is the connected subgroup of G whose Lie algebra is the span of all components of $r \in \mathrm{Lie}\, G \wedge \mathrm{Lie}\, G$, recall (3.1). Then all double cosets PtP, $t \in G$ are complete Poisson submanifolds of (G, π_r).*

Proof. We apply Lemma 3.2 to the decomposition

$$G = \sqcup_{[t] \in P \backslash G / P} PtP$$

over some representatives of all (P, P)-double cosets of G. Observe that

$$\pi_{r\, x} \in \wedge^2 T_x(PtP) \ \forall t \in G, x \in PtP$$

because of the definition (1.1) of π_r and the fact that the left and right-invariant bivector fields $\mathcal{L}_r$ and $\mathcal{R}_r$ on G are tangent to PtP for all $t \in G$. □

In the rest of this section we describe how each of the Poisson submanifolds PtP of (G, π_r) is obtained as a quotient of a symplectic manifold in the framework of Proposition 2.3. In particular we complete the proof of Theorem 1.1. Consider the left-invariant Poisson structure $(-\mathcal{L}_r, \mathcal{L}_r)$ on $\widetilde{P} \times \widetilde{P}$ coming from the skew-symmetric solution $(-r, r)$ of the classical Yang–Baxter on $\mathfrak{p} \oplus \mathfrak{p}$. Recall from the Introduction the definition of the subgroup

$$\widetilde{P}_t = \{(p_1, p_2) \in \widetilde{P} \times \widetilde{P} \mid \eta(p_2) = \mathrm{Ad}_t \eta(p_1)\} \subset \widetilde{P} \times \widetilde{P}. \tag{3.5}$$

It is clear that $\widetilde{P}_t$ is a closed subgroup of $\widetilde{P} \times \widetilde{P}$. Observe that the fibers of the projection

$$\nu_t\colon \widetilde{P} \times \widetilde{P} \to PtP, \quad \nu_t(p_1, p_2) = \eta(p_1^{-1}) t \eta(p_2), \ p_i \in \widetilde{P} \tag{3.6}$$

are exactly the left $\widetilde{P}_t$ orbits on $\widetilde{P} \times \widetilde{P}$. It is easy to see that

$$\nu_{t*}(-\mathcal{L}_r, \mathcal{L}_r) = -\mathcal{R}_r + \mathcal{L}_r = \pi_r$$

using the fact that for any triangular Poisson Lie group (G, π_r) we have an isomorphism of Poisson manifolds $(G, \mathcal{L}_r) \to (G, \mathcal{R}_r)$ given by $g \mapsto g^{-1}$. Thus $(PtP, \pi_r|_{PtP})$ is the quotient of $(\widetilde{P} \times \widetilde{P}, (-\mathcal{L}_r, \mathcal{L}_r))$ by the left action of the subgroup $\widetilde{P}_t$ of $\widetilde{P} \times \widetilde{P}$. To apply Proposition 2.4 we need the explicit form of the corresponding moment maps. Let

$$\mathfrak{p}_t = \mathfrak{p} \cap \mathrm{Ad}_t(\mathfrak{p}).$$

We will identify the Lie algebra of $\widetilde{P}$ with $\mathfrak{p}_t$ by the isomorphism

$$\mathrm{Lie}\,(\widetilde{P}_t) = \{(x, \mathrm{Ad}_t(x)) \mid x \in \mathfrak{p}_t\} \cong \mathfrak{p}_t$$

given by the projection into the first factor.

Recall that $\mathcal{B}\colon \widetilde{P} \to \mathfrak{p}^*$ is a group 1-cocycle with $d_e\mathcal{B} = \check{r}^{-1}$. Thus the map $(p_1, p_2) \mapsto (-\mathcal{B}(p_1), \mathcal{B}(p_2)) \in \mathfrak{p}^* \oplus \mathfrak{p}^*$ is a 1-cocycle for $\widetilde{P} \times \widetilde{P}$ and according to Proposition 2.2 it is a moment map for the left action of $\widetilde{P}\times\widetilde{P}$ on $(\widetilde{P}\times\widetilde{P}, (-\mathcal{L}_r, \mathcal{L}_r))$. If

$$\text{(3.7)} \quad \mathrm{pr}'_t\colon \mathfrak{p}^* \to \mathfrak{p}_t^* = (\mathfrak{p} \cap \mathrm{Ad}_t(\mathfrak{p}))^* \text{ and } \mathrm{pr}''_t\colon \mathfrak{p}^* \to (\mathrm{Ad}_t^{-1}\mathfrak{p}_t)^* = (\mathrm{Ad}_t^{-1}(\mathfrak{p}) \cap \mathfrak{p})^*$$

denote the natural projections, then the map $(p_1, p_2) \mapsto (-\mathrm{pr}'_t\mathcal{B}(p_1), \mathrm{pr}''_t\mathcal{B}(p_2)) \in \mathfrak{p}_t^* \oplus (\mathrm{Ad}_t^{-1}\mathfrak{p}_t)^*$ is a moment map for the left action of $\eta^{-1}(P \cap \mathrm{Ad}_t P) \times \eta^{-1}(\mathrm{Ad}_t^{-1}P \cap P)$ on $(\widetilde{P} \times \widetilde{P}, (-\mathcal{L}_r, \mathcal{L}_r))$. This finally means that the map

$$\text{(3.8)} \qquad J_t(p_1, p_2) = -\mathrm{pr}'_t\mathcal{B}(p_1) + \mathrm{Ad}^*_{t^{-1}}\mathrm{pr}''_t\mathcal{B}(p_2), \quad J_t\colon \widetilde{P} \times \widetilde{P} \to \mathfrak{p}_t^*$$

is a moment map for the left action of $\widetilde{P}_t$ on $(\widetilde{P} \times \widetilde{P}, (-\mathcal{L}_r, \mathcal{L}_r))$. Eq. (3.8) is just another way of writing the moment map $J_t(.,.)$ from (1.5). Now Theorem 1.1 follows from Proposition 2.3.

Theorem 1.1 simplifies in two special cases.

Corollary 3.4. *Assume that π_r is a triangular Poisson structure on a Lie group P such that $r \in \mathfrak{p}\wedge\mathfrak{p}$ is nondegenerate (i.e. r corresponds to a quasi-Frobenius structure on $\mathfrak{p}$). If the 1-cocycle $\check{r}^{-1}\colon \mathfrak{p} \to \mathfrak{p}^*$ lifts to a group cocycle $\mathcal{B}\colon P \to \mathfrak{p}^*$, then the symplectic leaves of (P, π_r) are the connected components of $\mathcal{B}^{-1}(\mathcal{O})$ for $\mathcal{O}$ a coadjoint orbit in $\mathfrak{p}^*$.*

To deduce Corollary 3.4 from Theorem 1.1 observe that $P = PeP$, P_e is the diagonal of $P \times P$, and $\mathcal{B}_t\colon P_e \to \mathfrak{p}^*$ vanishes because $\mathcal{B}_t(p, p) = -\mathcal{B}(p) + \mathcal{B}(p) = 0$ for all $p \in P$. In addition

$$J_e(p_1, p_2) = -\mathcal{B}(p_1) + \mathcal{B}(p_2) = \mathrm{Ad}^*_{p_1}(\mathcal{B}(p_1^{-1}) + \mathrm{Ad}^*_{p_1^{-1}}\mathcal{B}(p_2)) = \mathrm{Ad}^*_{p_1}\mathcal{B}(p_1^{-1}p_2),$$

for all pairs $(p_1, p_2) \in P \times P$.

Corollary 3.4 was independently obtained by Diatta and Medina in [**DM**].

Secondly Theorem 1.1 also simplifies in the case when $\mathfrak{r} \in \mathfrak{g} \wedge \mathfrak{g}$ is induced by a Frobenius structure on a subalgebra of $\mathfrak{g}$. We will omit the proof of the following result which is easily obtained from Proposition 2.4.

Proposition 3.5. *Assume that (G, π_r) is a triangular Poisson Lie group for which $r \in \mathfrak{p} \wedge \mathfrak{p}$, recall* (3.1), *is induces by a Frobenius structure $\xi \in \mathfrak{p}^*$. Then the symplectic leaves of each Poisson submanifold PtP, $(\forall t \in G)$ are the connected components of $\nu_t(I_t^{-1}(\mathcal{O}))$ where $\mathcal{O}$ is a coadjoint orbit of $\widetilde{P}_t$ and*

$$I_t\colon \widetilde{P}\times\widetilde{P} \to \mathfrak{p}_t^* \cong \mathfrak{g}^*/\mathfrak{p}_t^\perp \;\text{ is given by }\; I_t(p_1, p_2) = -\mathrm{Ad}^*_{p_1}(\overline{\xi}) + \mathrm{Ad}^*_{t^{-1}p_2}(\overline{\xi}) + \mathfrak{p}_t^\perp, p_i \in \widetilde{P}.$$

Here $\overline{\xi}$ is an arbitrary preimage of ξ under the projection $\mathfrak{g}^* \to \mathfrak{p}^*$. Only the coadjoint orbits in $\mathrm{pr}'_t(\mathrm{Ad}^*_P(\xi)) + \mathrm{Ad}^*_{t^{-1}}\mathrm{pr}''_t(\mathrm{Ad}^*_P(\xi))$ have nontrivial preimages, cf. (3.7).

Equivalently, the symplectic leaf of $(PgP, \pi_r|_{PgP})$ through PgP is the connected component of the submanifold

$$Y_g = \{p_1^{-1}gp_2 \mid p_i \in \widetilde{P},\ (\mathrm{Ad}^*_{p_1}(\overline{\xi}) - \overline{\xi}) - \mathrm{Ad}^*_{t^{-1}}(\mathrm{Ad}^*_{p_2}(\overline{\xi}) - \overline{\xi}) \in \mathfrak{p}_t^\perp\},$$

containing PgP.

4. Triangular Poisson structures induced from parabolic subgroups

Some of the most interesting examples of triangular Poisson structures come from Frobenius parabolic subalgebras of complex simple Lie algebras, see [**GG**].

Assume that G is a complex simple Lie group and that P is a parabolic subgroup whose Lie algebra $\mathfrak{p}$ has a Frobenius structure $\xi_0 \in \mathfrak{p}^*$. It gives rise to a nondegenerate r-matrix $r \in \mathfrak{p} \wedge \mathfrak{p} \subset \mathfrak{g} \wedge \mathfrak{g}$, see Section 2. In this section we study the corresponding Poisson structure π_r on G.

Denote by W_L the Weyl group of the pair (L, T) where L is a Levi factor of P and T is a maximal torus of L. Denote by W the Weyl group of the pair (G, T). The Bruhat decomposition of G with respect to the parabolic subgroup P is

$$G = \bigsqcup_{w \in W_L \backslash W / W_L} PwP.$$

Recall that every double coset $W_L \backslash W / W_L$ has a unique element of a minimal length, see e.g. [**C**]. The set of such is usually denoted by ${}^{W_L}W^{W_L}$ and is identified with $W_L \backslash W / W_L$. Proposition 3.5 now implies the following description of the symplectic leaves of (G, π_r).

Corollary 4.1. *In the above setting the Poisson Lie group (G, π_r) decomposes into the complete Poisson submanifolds PwP for $w \in {}^{W_L}W^{W_L}$. If $\nu_w \colon P \times P \to PwP$ denotes the projection $\nu_w(p_1, p_2) = p_1^{-1} w p_2$ then the symplectic leaves of PwP, are the connected components of $\nu_t(I_t^{-1}(\mathcal{O}))$ where $\mathcal{O}$ is a coadjoint orbit of $\widetilde{P}_t$ and*

$I_t \colon \widetilde{P} \times \widetilde{P} \to \mathfrak{p}_t^* \cong \mathfrak{g}^*/\mathfrak{p}_t^\perp$ *is given by* $I_t(p_1, p_2) = -\mathrm{Ad}^*_{p_1}(\overline{\xi}) + \mathrm{Ad}^*_{t^{-1}p_2}(\overline{\xi}) + \mathfrak{p}_t^\perp, p_i \in \widetilde{P}$.

Here $\overline{\xi}$ is an arbitrary preimage of ξ under the projection $\mathfrak{g}^* \to \mathfrak{p}^*$.

The simplest examples of this type are Gerstenhaber–Giaquinto's [**GG**] generalized Jordanian Poisson structures on $SL(n)$, induced from the r-matrices

$$r = n \sum_{i<j} \sum_{k=i}^{j-1} E_{k,i} \wedge E_{i+j-k-1,j} + \sum_{i,j} (j-1) E_{j-1,j} \wedge E_{i,i}.$$

Such an r-matrix corresponds to the maximal parabolic subalgebra of $\mathfrak{sl}(n)$, associated to an extremal root

$$\mathfrak{p} = \mathrm{Span}\{E_{i,j} \mid i < n, j \leq n\},$$

and is induced by the following Frobenius structure on $\mathfrak{p}$, [**GG**, Theorem 5.9]

$$\xi = -n \sum_{i=1}^{n-1} E^*_{i\ i+1}.$$

Here $\{E^*_{ij}\}_{i,j=1}^n$ denotes the dual basis to the basis $\{E_{ij}\}_{i,j=1}^n$ of the space of $n \times n$ matrices. The Weyl group of $SL(n)$ is identified with S_n and its subgroup corresponding to the Levi subalgebra of $\mathfrak{p}$, containing the Cartan subalgebra of diagonal matrices, is S_{n-1}. The related minimal length elements, parametrizing the Poisson submanifolds in Corollary 4.1, are

$${}^{S_{n-1}}S_n^{S_{n-1}} = \{\mathrm{id}, (n-1\ n)\}.$$

The subalgebra $\mathfrak{p}_{(n-1\ n)}$ is easily described:

$$\mathfrak{p}_{(n-1\ n)} = \mathrm{Span}\{E_{i,j} \mid i < n, j \leq n\}.$$

It does not depend on the representative of $(n-1\ n) \in S_n$ in the normalizer of T.

In the rest of this section we provide full details for the case $n = 2$. We will use the standard basis $\{E, F, H\}$ of $\mathfrak{sl}(2)$ and the corresponding dual basis $\{E^*, F^*, H^*\}$ of $\mathfrak{sl}(2)^*$. The Jordanian Poisson structure on $G = SL(2)$ corresponds to the r-matrix $r = E \wedge H$. The latter is induced from the Borel subalgebra $\mathfrak{b}$ generated by E and H. If we identify E^* and H^* with their images in $\mathfrak{b}^*$, then the Frobenius structure on $\mathfrak{b}^*$ related to r is simply

$$\xi = -E^*.$$

The cocycle $\check{r}^{-1}\colon \mathfrak{b} \to \mathfrak{b}^*$ integrates to the group cocycle

$$\mathcal{B}(g) = E^* - \mathrm{Ad}^*_g(E^*) = (1-a^{-2})E^* - 2a^{-1}bH^* \in \mathfrak{b}^*, \quad g = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in B.$$

where B denotes the Borel subgroup of $SL(2)$, consisting of upper triangular matrices. The decompositions of $(SL(2), \pi_r)$ into complete Poisson submanifolds from Corollary 4.1 is just the Bruhat decomposition $G = B \cup BsB$. Here s is a representative of the nontrivial element of the Weyl group in the normalizer of the maximal torus T of diagonal matrices, say $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The symplectic leaves of $(SL(2), \pi_r)$ are then of two types – "s" and "e". The leaves of type e are the inverse images $\mathcal{B}^{-1}(\mathcal{O})$ for $\mathcal{O}$ a coadjoint orbit of $\mathfrak{b}^*$. The coadjoint orbits of $\mathfrak{b}^*$ are the points αH^* and the complement to the set of these points. The former give rise to the symplectic points

$$\left\{ \pm \begin{bmatrix} 1 & -\alpha/2 \\ 0 & 1 \end{bmatrix} \right\}.$$

If S denotes the set of such points, the inverse image of the 2 dimensional coadjoint orbit in $\mathfrak{b}^*$ is the complement of S in B which is a single symplectic leaf.

Now consider the case when $t = s$. Then $\mathfrak{b}_s = \mathfrak{b} \cap \mathrm{Ad}_s(\mathfrak{b}) = \mathfrak{h} = \mathrm{Lie}\, T$ and $I_s\colon B \times B \to \mathfrak{h}$ is given by

$$I_s(g_1, g_2) = \mathrm{Ad}^*_{g_1}(E^*) - \mathrm{Ad}^*_{sg_2}(E^*) + \mathfrak{h}^\perp = 2(a_1^{-1}b_1 + a_2^{-1}b_2)H^*, \quad g_i = \begin{bmatrix} a_i & b_i \\ 0 & a_i^{-1} \end{bmatrix}.$$

The coadjoint orbits of $G_s = T$ are just the points αH^*. Thus the symplectic leaves of BsB are the connected components of

$$\nu_s(I_s^{-1}(\alpha H^*)) = \{g_1^{-1} s g_2 \mid 2(a_1^{-1}b_1 + a_2^{-1}b_2) = \alpha\}.$$

The latter is easily seen to be the locally closed variety $\mathcal{V}(2(T_{11} - T_{22}) + \alpha T_{21}) - B$ which is connected. By T_{ij} we denote the standard coordinate functions on $SL(n)$.

Thus from an algebro-geometric point of view, the leaves can be described as follows. The zero dimensional symplectic leaves are the points of the variety $Z = \mathcal{V}(T_{11}^2 - 1, T_{21})$. The two dimensional leaves are the locally closed varieties $L_{[\lambda,\mu]} = \mathcal{V}(\lambda(T_{11} - T_{22}) - \mu T_{21}) - Z$ where $[\lambda, \mu] \in \mathbb{P}^1(\mathbb{C})$. Note that the closures of the two dimensional leaves all contain Z and that the intersection of the closures of any two distinct two-dimensional leaves is precisely Z.

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