Curvature Estimates for the Ricci Flow I

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1 Introduction

In this paper we present several curvature estimates for solutions of the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric$$

and its normalized versions, such as the volume normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric + \frac{2}{n}\mathcal{S}g \tag{1.1}$$

with $\mathcal{S}$ denoting the total scalar curvature and n denoting the dimension of the manifold, and the λ-normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric + \lambda g \tag{1.2}$$

with a constant λ. (Since Einstein metrics are stationary solutions of the volume normalized Ricci flow, they are included as a special case.) These estimates depend on the smallness of certain local $L^{\frac{n}{2}}$ integrals of the norm of the Riemann curvature tensor. A key common property of these integrals is scaling invariance, thanks to the critical exponent $\frac{n}{2}$. Because of this property, they are very natural and contain particularly rich geometric information. (Note that in dimension 4, the norm square of the Riemann curvature tensor is closely related to the Gauss-Bonnet-Chern integrand.)

To formulate our results, we need some terminologies. Consider a connected Riemannian manifold (M, g) (g denotes the metric) possibly with boundary, and $x \in M$. If x is in the interior of M, we define the distance $d(x, \partial M)$ to be $\sup\{r > 0 : B(x,r)$ is compact and contained in the interior of $M\}$, where $B(x,r)$ denotes the closed geodesic ball of center x and radius r. If M has a boundary and $x \in \partial M$, then $d(x, \partial M)$ is the ordinary distance from x to ∂M and equals zero. (For example, $d(x, \partial M) = \infty$ if M is closed.)

Let $g = g(t)$ be a family of metrics on M. Then $d(x, y, t)$ denotes the distance between $x, y \in M$ with respect to the metric $g(t)$, and $B(x, r, t)$ denotes the closed geodesic ball of center $x \in M$ and radius r with respect to the metric $g(t)$.

For a family $U(t), 0 \le t < T$ of open sets of M for some $T > 0$, we define its direct limit $\underline{\lim}_{t \to T} U(t)$ as follows. A point x of M lies in $\underline{\lim}_{t \to T} U(t)$, if there is a neighborhood U of x and some $t \in [0, T)$ such that $U \subset U(t')$ for all $t' \in [t, T)$.

We set $\alpha_n = \frac{1}{40(n-1)}$, $\epsilon_0 = \frac{1}{84}$ and $\epsilon_1 = \frac{\epsilon_0}{8\sqrt{1+2\alpha_n \epsilon_0^2}}$. (These constants are not meant to be optimal. One can improve them by carefully examing the proofs.)

Our first result involves the concept of κ-noncollapsedness due to Perelman. By [Theorem 4.1, P], a smooth solution of the Ricci flow on $M \times [0, T]$ for a closed manifold M and a finite T is κ-noncollapsed on the scale $\sqrt{T}$, where κ depends on the initial metric and T.

Theorem A *For each positive number κ and each natural number $n \ge 3$ there are positive constants $\delta_0 = \delta_0(\kappa, n)$ and $C_0 = C_0(n, \kappa)$ depending only on κ and n with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow or the volume normalized Ricci flow on $M \times [0, T)$ for a connected manifold M of dimension $n \ge 3$ and some (finite or infinite) $T > 0$, which is κ-noncollapsed on the scale of ρ for some $\kappa > 0$ and $\rho > 0$. Consider $x_0 \in M$ and $0 < r_0 \le \rho$, which satisfy $r_0 \le diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T)$. Assume that*

$$\int_{B(x_0, r_0, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dvol_{g(t)} \le \delta_0 \tag{1.3}$$

for all $t \in [0, T)$. Then we have

$$|Rm|(x, t) \le \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2} \tag{1.4}$$

whenever $t \in (0, T)$ and $d(x_0, x, t) < \epsilon_0 r_0$, and

$$|Rm|(x, t) \le C_0 \max\{r_0^{-2}, t^{-1}\} \sup_{0 \le t < T} \left(\int_{B(x_0, r_0, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dvol_{g(t)}\right)^{\frac{2}{n}} \tag{1.5}$$

whenever $0 < t < T$ and $d(x_0, x, t) \le \epsilon_1 \min\{r_0, \sqrt{t}\}$. (Obviously, it follows that if the assumptions hold on $[0, T]$, then the estimates (1.4) and (1.5) hold on $[0, T]$. This remark also applies the the results below.)

If T is finite, then $g(t)$ converges smoothly to a smooth metric $g(T)$ on $\underline{\lim}_{t \to T} \overset{\circ}{B}(x_0, \epsilon_0 r_0, t)$. Moreover, $\overset{\circ}{B}(x_0, \epsilon_0 r_0, T) = \underline{\lim}_{t \to T} \overset{\circ}{B}(x_0, \epsilon_0 r_0, t)$. (Here and below, smooth convergence means smooth convergence on each compact subset.) If $T = \infty$, then we have similar smooth subconvergence of $g(t)$ as $t \to T$.

The same results hold for the λ-normalized Ricci flow, with δ_0 and C_0 also depending on $|\lambda|$.

Corollary A *Let $g = g(t)$ be a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow on $M \times [0, T)$ for an n-dimensional connected manifold M and some finite $T > 0$, such that $g(t)$ is complete for each $t \in [0, T)$. There is a positive constant $\delta_0 = \delta_0(n, T, g(0))$ depending only on n, T and $g(0)$ with the following property. Assume that (1.3) holds true for all $x_0 \in M$, all $t \in [0, T)$, and some r_0 satisfying $r_0 \le T^{\frac{1}{2}}$ and $r_0 \le diam_{g(t)}(M)$ for all $t \in [0, T)$. Then (1.4) holds for all $x \in M$ and $t \in [0, T)$. Consequently, $g(t)$ extends to a smooth solution of the Ricci flow over $[0, T']$ for some $T' > T$.*

Our second result does not use the condition of κ-noncollapsedness. Instead, smallness of $L^{\frac{n}{2}}$ integrals of Rm over balls of varying center and radius measured against a volume ratio is assumed.

Theorem B *For each natural number $n \ge 3$ there are positive constants $\delta_0 = \delta_0(n)$ and $C_0 = C_0(n)$ depending only on n with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow or the volume normalized Ricci flow on $M \times [0, T)$ for a connected manifold M of dimension $n \ge 3$ and some (finite or infinite) $T > 0$. Consider $x_0 \in M$ and $r_0 > 0$,which satisfy $r_0 \le diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T)$. Assume that*

$$\int_{B(x,r,t)} |Rm|^{\frac{n}{2}}(\cdot, t) dvol_{g(t)} \le \delta_0 \frac{vol_{g(t)}(B(x, r, t))}{r^n} \tag{1.6}$$

whenever $t \in [0, T), 0 < r \le \frac{r_0}{2}$ and $x \in B(x_0, \frac{r_0}{2}, t)$. Then we have

$$|Rm|(x, t) \le \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2} \tag{1.7}$$

whenever $t \in (0, T)$ and $d(x_0, x, t) < \epsilon_0 r_0$, and

$$|Rm|(x, t) \le C_0 \sup_{0 \le t < T} \left(\frac{\int_{B(x_0, 2r_1, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dvol_{g(t)}}{vol_{g(t)}(B(x_0, 2r_1, t))} \right)^{\frac{2}{n}} \tag{1.8}$$

whenever $0 < t < T$ and $d(x_0, x, t) \le r_1$, where $r_1 = \epsilon_1 \min\{r_0, \sqrt{t}\}$.

If T is finite, then $g(t)$ converges smoothly to a smooth metric $g(T)$ on $\underline{\lim}_{t \to T} \mathring{B}(x_0, \epsilon_0 r_0, t)$. Moreover, $\mathring{B}(x_0, \epsilon_0 r_0, T) = \underline{\lim}_{t \to T} \mathring{B}(x_0, \epsilon_0 r_0, t)$. If $T = \infty$, then we have similar smooth subconvergence of $g(t)$ as $t \to T$.

The same results hold for the λ-normalized Ricci flow, with δ_0 and C_0 also depending on $|\lambda|$.

Corollary B *Let $g = g(t)$ be a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow on $M \times [0, T)$ for an n-dimensional connected manifold M and some (finite or infinite) $T > 0$,such that $g(t)$ is complete* *for each $t \in [0, T)$. Assume that (1.6) holds true for all $x_0 \in M$, all $t \in [0, T)$, and all $0 < r \leq r_0$ for some positive number r_0 satisfying $r_0 \leq diam_{g(t)}(M)$ for all $t \in [0, T)$. Then (1.7) and (1.8) hold for all $x \in M$ and $t \in [0, T)$. Consequently, $g(t)$ extends to a smooth solution of the Ricci flow over $[0, T']$ for some $T' > T$ if T is finite. If $T = \infty$, then $g(t)$ subconverges smoothly as $t \to T$.*

Our third result does not use the condition of κ-noncollapsedness, and involves only a fixed center and a fixed radius for $L^{\frac{n}{2}}$ integrals of the norm of the Riemann curvature tensor. But a lower bound for the Ricci curvature is assumed.

Theorem C *For each natural number $n \geq 3$ there are positive constants $\delta_0 = \delta_0(n)$ and $C_0 = C_0(n)$ depending only on n with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow or the normalized Ricci flow on $M \times [0, T)$ for a connected manifold of dimension $n \geq 3$ and some (finite or infinite) $T > 0$. Consider $x_0 \in M$ and $r > 0$, which satisfy $r_0 \leq diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T]$. Assume that*

$$Ric(x,t) \geq -\frac{n-1}{r_0^2} g(x,t) \tag{1.9}$$

whenever $t \in [0, T)$ and $d(x_0, x, t) \leq r_0$ ($g(x,t) = g(t)(x)$ and $Ric(x,t)$ is the Ricci tensor of $g(t)$ at x), and that

$$\int_{B(x_0, r_0, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dvol_{g(t)} \leq \delta_0 \frac{vol_{g(t)}(B(x_0, r_0, t))}{r_0^n} \tag{1.10}$$

for all $t \in [0, T]$. Then we have

$$|Rm|(x,t) \leq \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2} \tag{1.11}$$

whenever $t \in (0, T)$ and $d(x_0, x, t) < \epsilon_0 r_0$, and

$$|Rm|(x,t) \leq C_0 \sup_{0 \leq t < T} \left(\frac{\int_{B(x_0, 2r_1, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dvol_{g(t)}}{vol_{g(t)}(B(x_0, 2r_1, t))} \right)^{\frac{2}{n}} \tag{1.12}$$

whenever $0 < t < T$ and $d(x_0, x, t) \leq r_1$, where $r_1 = \epsilon_1 \min\{r_0, \sqrt{t}\}$.

If T is finite, then $g(t)$ converges smoothly to a smooth metric $g(T)$ on $\underline{\lim}_{t \to T} \overset{\circ}{B}(x_0, \epsilon_0 r_0, t)$. Moreover, $\overset{\circ}{B}(x_0, \epsilon_0 r_0, T) = \underline{\lim}_{t \to T} \overset{\circ}{B}(x_0, \epsilon_0 r_0, t)$. If $T = \infty$, then we have similar smooth subconvergence of $g(t)$ as $t \to T$.

The same results hold for the λ-normalized Ricci flow, with δ_0 and C_0 also depending on $|\lambda|$.

Note that an elliptic analogue of (1.12) for Einstein metrics can be found in [An].

Corollary C *Let $g = g(t)$ be a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow on $M \times [0,T)$ for an n-dimensional connected manifold M and some (finite or infinite) $T > 0$, such that $g(t)$ is complete for each $t \in [0,T)$. Assume that (1.9) and (1.10) hold true for all $x_0 \in M$ and some positive number r_0 satisfying $r_0 \le diam_{g(t)}(M)$ for all $t \in [0,T)$. Then (1.11) and (1.12) holds for all $x \in M$ and $t \in [0,T)$. Consequently, $g(t)$ extends to a smooth solution of the Ricci flow over $[0,T']$ for some $T' > T$, if T is finite. If $T = \infty$, then $g(t)$ subconverges smoothly as $t \to T$.*

The above curvature estimates can be used to deduce convergence results under the condition of finite $L^{\frac{n}{2}}$ integrals of the norm of the Riemann curvature tensor. This will be presented elsewhere.

Similar results also hold for many other evolution equations. This will be presented elsewhere.

The results in this paper were obtained some time ago.

Analogous results involving other types of L^p integrals of $|Rm|$, including the case $p < \frac{n}{2}$ and space-time integrals, will be presented in sequels of this paper.

2 A Linear Parabolic Estimate

In this section we present a linear parabolic estimate based on Moser's iteration, which will be needed for establishing our curvature estimates. First we fix some notations. Consider a Riemannian manifold (M,g) of dimension n. The Sobolev constant $C_{S,g}(M)$ is defined to be the smallest number $C_{S,g}(M)$ such that

$$\|f\|_{\frac{n}{n-1}} \le C_{S,g}(M)\|\nabla f\|_1 \tag{2.1}$$

for all Lipschitz functions f on M with compact support, where $\|\cdot\|_p$ means the L^p-norm. More precisely,

$$C_{S,g}(M) = \sup\{\|f\|_{\frac{n}{n-1}} : f \in C^1_c(M), \|\nabla f\|_1 = 1\}.$$

$C_{S,g}(M)$ equals the isoperimetric constant $C_{I,g}(M)$ of (M,g), which is defined to be $\sup\{\frac{vol(\Omega)^{\frac{n-1}{n}}}{vol(\partial\Omega)} : \Omega \subset M$ is a C^1 domain with compact closure.$\}$ The L^2-Sobolev constant $C_{S,2,g}(M)$ is defined to be the smallest number $C_{S,2,g}(M)$ such that

$$\|f\|_{\frac{2n}{n-2}} \le C_{S,2,g}(M)\|\nabla f\|_2.$$

Note that $C_{S,2,g}(M) \le \frac{2(n-1)}{n-2} C_{S,g}(M)$. Indeed, applying (2.1) to $|f|^{\frac{2(n-1)}{n-2}}$ we deduce

$$\begin{aligned}(\int_M |f|^{\frac{2n}{n-2}} dvol_g)^{\frac{n-1}{n}} &\le C_{S,g}(M) \int |\nabla |f|^{\frac{2(n-1)}{n-2}}| dvol_g \\ &\le C_{S,g}(M) \frac{2(n-1)}{n-2} \|\nabla f\|_2 (\int_M |f|^{\frac{2n}{n-2}} dvol_g)^{\frac{1}{2}}.\end{aligned}$$

Hence the claimed inequality follows.

The following result is taken from [Ye2]. We include the proof for the convenience of the reader, and for the reason of verifying the explicit dependence on the Sobolev constant, which is important for the curvature estimates in this paper.

Theorem 2.1 *Let M be a smooth manifold of dimension n and $g = g(t)$ a smooth family of Riemannian metrics on M for $t \in [0,T]$. Let f be a nonnegative Lipschitz continuous function on $M \times [0,T]$ satisfying*

$$\frac{\partial f}{\partial t} \leq \Delta f + af \tag{2.2}$$

on $M \times [0,T]$ in the weak sense, where a is a nonnegative constant and $\Delta = \Delta_{g(t)}$. Let x_0 be an interior point of M. Then we have for each $p_0 > 1$ and $0 < R < d_{g(0)}(x_0, \partial M)$

$$\begin{aligned} |f(x,t)| \;&\leq\; (1+\frac{2}{n})^{\frac{\sigma_n}{p_0}} C_{S,2}^{\frac{n}{p_0}} \left(ap_0 + \frac{\gamma}{2} + \frac{n}{2}(1+\frac{n}{2})^2 \cdot \frac{1}{t} + \frac{(n+2)^2 e^{-\lambda_* T}}{4R^2} \right)^{\frac{n+2}{2p_0}} \\ &\cdot\; \Big(\int_0^T \int_{B(x_0,R,0)} f^{p_0}(\cdot,t) dvol_{g(t)} dt \Big)^{\frac{1}{p_0}} \;, \end{aligned} \tag{2.3}$$

whenever $0 < t \leq T$ and $d_{g(0)}(x_0, x) \leq \frac{R}{2}$, where $\sigma_n = \sum_0^\infty \frac{2k}{(1+\frac{2}{n})^k}$, γ denotes the maximum value of the trace of $\frac{\partial g}{\partial t}$ on $B(x_0,R,0) \times [0,T]$, λ_* denotes the minimum eigenvalue of $\frac{\partial g}{\partial t}$ on $B(x_0,R,0) \times [0,T]$, and

$$C_{S,2} = \max_{0 \leq t \leq T} C_{S,g(t),2}(B(x_0,R,0)).$$

The same estimate holds if we replace $B(x_0,R,0)$ by $B(x_0,R,T)$, and $-\lambda_*$ by λ^*, which denotes the maximum eigenvalue of $\frac{\partial g}{\partial t}$ on $B(x_0,R,T) \times [0,T]$.

Proof. We handle the case of $B(x_0,R,0)$, while the other case is similar. Let η be a non-negative Lipschitz function on M whose support is contained in $B(x_0,R,0)$. The partial differential inequality (2.2) implies for $p \geq 2$

$$\frac{1}{p}\frac{\partial}{\partial t}\int f^p \eta^2 dvol_{g(t)} \leq -\int \nabla(\eta^2 f^{p-1}) \cdot \nabla f \, dvol_{g(t)} + \int b f^p \eta^2 dvol_{g(t)} + \frac{1}{p}\int f^p \eta^2 \frac{\partial}{\partial t} dvol_{g(t)} \,.$$

We'll omit the notation $dvol_{g(t)}$ below. We have

$$\begin{aligned} -\int \nabla(\eta^2 f^{p-1}) \cdot \nabla f \;&=\; -\frac{4(p-1)}{p^2}\int |\nabla(\eta f^{p/2})|^2 + \frac{4}{p^2}\int |\nabla\eta|^2 f^p \\ &\quad + \frac{4(p-2)}{p^2}\int \nabla(\eta f^{p/2}) f^{p/2} \nabla\eta \\ &\leq\; -\frac{2}{p}\int |\nabla(\eta f^{p/2})|^2 + \frac{2}{p}\int |\nabla\eta|^2 f^p, \end{aligned}$$

where $\nabla = \nabla_{g(t)}$. Therefore

$$\frac{\partial}{\partial t}\int f^p\eta^2 + 2\int |\nabla(\eta f^{p/2})|^2 \leq 2\int |\nabla\eta|^2 f^p + (pa + \frac{\gamma}{2})\int f^p\eta^2 \ . \tag{2.4}$$

Next we define for $0 < \tau < \tau' < T$

$$\psi(t) = \begin{cases} 0 & 0 \leq t \leq \tau \ , \\ (t-\tau)/(\tau'-\tau) & \tau \leq t \leq \tau' \ , \\ 1 & \tau' \leq t \leq T \ . \end{cases}$$

Multiplying (2.4) by ψ, we obtain

$$\frac{\partial}{\partial t}\left(\psi\int f^p\eta^2\right) + 2\psi\int |\nabla(\eta f^{p/2})|^2 \leq 2\psi\int |\nabla\eta|^2 f^p + ((pa + \frac{\gamma}{2})\psi + \psi')\int f^p\eta^2 \ .$$

Integrating this with respect to t we get

$$\int_t f^p\eta^2 + 2\int_{\tau'}^t\int |\nabla(\eta f^{p/2})|^2 \leq 2\int_\tau^T\int |\nabla\eta|^2 f^p + \left(pa + \frac{\gamma}{2} + \frac{1}{\tau'-\tau}\right)\int_\tau^T\int f^p\eta^2$$

for $\tau' \leq t \leq T$. Applying this estimate and the Sobolev inequality we deduce

$$\begin{aligned} \int_{\tau'}^T\int f^{p(1+\frac{2}{n})}\eta^{2+\frac{1}{n}} &\leq \int_{\tau'}^T\left(\int f^p\eta^2\right)^{2/n}\left(\int f^{\frac{pn}{n-2}}\eta^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq C_{S,2}^2\Big(\sup_{\tau'\leq t\leq T}\int f^p\eta^2\Big)^{2/n}\int_{\tau'}^T\int \left|\nabla(\eta f^{p/2})\right|^2 \\ &\leq C_{S,2}^2\left[2\int_\tau^T\int |\nabla\eta|^2 f^p + \left(pa + \frac{\gamma}{2} + \frac{1}{\tau'-\tau}\right)\int_\tau^T\int f^p\eta^2\right]^{1+\frac{2}{n}} \ . \end{aligned} \tag{2.5}$$

We put

$$H(p,\tau,R) = \int_\tau^T\int_{B(x_0,R,0)} f^p$$

for $0 < \tau < T$ and $0 < R < d_{g(0)}(x_0,\partial M)$. Given $0 < R' < R < d_{g(0)}(x_0,\partial M)$, we define $\eta(x) = 1$ for $d(x_0,x,0)\ leqR'$, $\eta(x) = 1 - \frac{1}{R-R'}(d(x_0,x,0) - R')$ for $R' \leq d(x_0,x,0) \leq R$, and $\eta(x) = 0$ for $d(x_0,x,0) \geq R$. Noticing $|\nabla\eta|_t \leq \frac{1}{R-R'}e^{-\frac{1}{2}\lambda_* t}$ we derive from (2.5)

$$H\left(p\left(1+\frac{2}{n}\right),\tau',R'\right) \leq C_S^2\left[pa + \frac{\gamma}{2} + \frac{1}{\tau'-\tau} + \frac{2e^{-\lambda_* T}}{(R-R')^2}\right]^{1+\frac{2}{n}} H(p,\tau,R)^{1+\frac{2}{n}} \ . \tag{2.6}$$

Now we fix $0 < R < d_{g(0)}(x_0, \partial M)$ and set $\mu = 1 + \frac{2}{n}$, $p_k = p_0 \mu^k$, $\tau_k = (1 - \frac{1}{\mu^{k+1}})t$ and $R_k = \frac{R}{2}(1 + \frac{1}{\mu^k})$ with $R = \frac{1}{2} dist_{g(0)}(x, \partial N)$. Then it follows from (2.6) that

$$
\begin{aligned}
&H(p_{k+1}, \tau_{k+1}, R_{k+1})^{\frac{1}{p_{k+1}}} \leq \\
&C_S^{\frac{2}{p_{k+1}}} \left[a p_k + \frac{\gamma}{2} + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + \frac{2e^{-\lambda_* T} \mu^2}{R^2(\mu - 1)^2} \right]^{\frac{1}{p_k}} \mu^{\frac{k}{p_k}} H(p_k, \tau_k, R_k)^{\frac{1}{p_k}} \leq \\
&\quad C_S^{\frac{2}{p_{k+1}}} \left[a p_0 + \frac{\gamma}{2} + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + \frac{2e^{-\lambda_* T} \mu^2}{R^2(\mu - 1)^2} \right]^{\frac{1}{p_k}} \mu^{\frac{2k}{p_k}} H(p_k, \tau_k, R_k)^{\frac{1}{p_k}} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
&H(p_{m+1}, \tau_{m+1}, R_{m+1})^{\frac{1}{p_{m+1}}} \leq C_S^{\sum_0^m \frac{2}{p_{k+1}}} \mu^{\sum_0^m \frac{2k}{p_k}} \\
&\cdot \left[a p_0 + \frac{\gamma}{2} + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + \frac{e^{-\lambda_* T} \mu^2}{R^2(\mu - 1)^2} \right]^{\sum_0^m \frac{1}{p_k}} H(p_0, \tau_0, R_0)^{\frac{1}{p_0}}.
\end{aligned}
$$

Letting $m \to \infty$ we arrive at (2.3). ■

3 Proof of Theorem A

For the convenience of the reader, we state Perelman's definition of κ-noncollapsedness.

Definition Let g be a Riemannian metric on a manifold M of dimension n. Let κ and ρ be positive numbers. We say that g is κ-noncollapsed, if there holds $vol_g(B(x,r)) \geq \kappa r^n$ for each geodesic ball $B(x,r)$ of (M,g) satisfying $\sup\{|Rm|(x) : x \in B(x,r)\} \leq r^{-2}$.

Proof of Theorem A

We present the case of the Ricci flow. The cases of the volume normalized Ricci flow and the λ-normalized Ricci flow are similar.

Proof of the estimate (1.4)

By rescaling, we can assume $r_0 = 1$. Assume that the estimate (1.4) does not hold. Then we can find for each $\epsilon > 0$ a Ricci flow solution $g = g(t)$ on $M \times [0,T]$ for some M and $T > 0$ with the properties as postulated in the statement of the theorem, such that $|Rm|(x,t) > \alpha_n t^{-1} + \epsilon^{-2}$ for some $(x,t) \in M \times [0,T]$ satisfying $d(x_0, x, t) < \epsilon$.

We denote by M_{α_n} the set of pairs (x,t) such that $|Rm|(x,t) \geq \alpha_n t^{-1}$. For an arbitary positive number $A > 1$ such that $(2A+1)\epsilon \leq \frac{1}{2}$, we choose as in [Proof of Theorem 10.1, P] $(\bar{x},\bar{t}) \in M_{\alpha_n}$ with $0 < \bar{t} \leq \epsilon^2, d(x_0,\bar{x},\bar{t}) < (2A+1)\epsilon$, such that $|Rm|(\bar{x},\bar{t}) > \alpha_n \bar{t}^{-1} + \epsilon^{-2}$ and

$$|Rm|(x,t) \leq 4|Rm|(\bar{x},\bar{t}) \tag{3.1}$$

whenever

$$(x,t) \in M_{\alpha_n}, 0 < t \leq \bar{t}, d(x_0,x,t) \leq d(x_0,\bar{x},\bar{t}) + A|Rm|(\bar{x},\bar{t})^{-\frac{1}{2}}. \tag{3.2}$$

We set $Q = |Rm|(\bar{x},\bar{t})$.

Claim 1 If

$$\bar{t} - \frac{1}{2}\alpha_n Q^{-1} \leq t \leq \bar{t}, d(\bar{x},x,\bar{t}) \leq \frac{1}{10}AQ^{-\frac{1}{2}}, \tag{3.3}$$

then

$$d(x_0,x,t) \leq d(x_0,\bar{x},\bar{t}) + \frac{1}{2}AQ^{-\frac{1}{2}}. \tag{3.4}$$

Note that (3.4) implies

$$d(x_0,x,t) \leq (2A+1)\epsilon + \frac{1}{2}AQ^{-\frac{1}{2}} \leq (\frac{5}{2}A+1)\epsilon \tag{3.5}$$

for (x,t) satisfying (3.3).

Proof of Claim 1

Since $(\bar{x},\bar{t}) \in M_{\alpha_n}$, we have $Q \geq \alpha_n \bar{t}^{-1}$, so

$$\bar{t} - \frac{1}{2}\alpha_n Q^{-1} \geq \frac{1}{2}\bar{t}. \tag{3.6}$$

Consider $(\hat{x},\hat{t})$ satisfying (3.3). By the triangular inequality, we have $d(x_0,\hat{x},\bar{t}) \leq d(x_0,\bar{x},\bar{t}) + \frac{1}{10}AQ^{-\frac{1}{2}}$. We estimate $d(x_0,\hat{x},\hat{t})$. For this purpose, consider the set I of $t^* \in [\hat{t},\bar{t}]$ such that

$$d(x_0,\hat{x},t) \leq d(x_0,\bar{x},\bar{t}) + \frac{1}{2}AQ^{-\frac{1}{2}} \tag{3.7}$$

for all $t \in [t^*,\bar{t}]$. Obviously, I is closed and $\bar{t} \in I$. We claim that it is open in $[\hat{t},\bar{t}]$. Consider $t^* \in I$. For each $t \in [t^*,\bar{t}]$, we follow [P] and apply [Lemma 8.3(b),P] to $x_0,\hat{x}$.

We set $R = \frac{1}{2}AQ^{-\frac{1}{2}}$. For $x \in B(x_0, R, t)$ we have $|Rm|(x,t) \leq 4Q$ if $(x,t) \in M_{\alpha_n}$. If $x \notin M_{\alpha_n}$, we have by (3.6)

$$|Rm|(x,t) \leq \alpha_n t^{-1} \leq 2\alpha \bar{t}^{-1} \leq 2Q. \tag{3.8}$$

For $x \in B(\hat{x}, R, t)$, we have $d(x_0, x, t) \leq d(x_0, \hat{x}, t) + d(\hat{x}, x, t) \leq d(x_0, \bar{x}, \bar{t}) + AQ^{-\frac{1}{2}}$. Hence $|Rm|(x,t) \leq 4Q$, if $(x,t) \in M_{\alpha_n}$. If $(x,t) \notin M_{\alpha_n}$, we obtain (3.8) as in the previous case. By [Lemma 8.3(b), P], we have

$$\frac{d}{dt} d(x_0, \hat{x}, t) \geq -2(n-1)(\frac{2}{3} \cdot 4Q \cdot \frac{1}{2}AQ^{-\frac{1}{2}} + 2A^{-1}Q^{\frac{1}{2}}) \geq -4(n-1)(A + \frac{1}{A})Q^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} d(x_0, \hat{x}, t^*) &\leq d(x_0, \hat{x}, \bar{t}) + \frac{1}{2}\alpha_n Q^{-1} \cdot 4(n-1)(A + \frac{1}{A})Q^{\frac{1}{2}} \\ &= d(x_0, \hat{x}, \bar{t}) + 2(n-1)\alpha(1 + \frac{1}{A^2})AQ^{-\frac{1}{2}}) \\ &\leq d(x_0, \bar{x}, \bar{t}) + \frac{1}{3}AQ^{-\frac{1}{2}}. \end{aligned}$$

By the continuity of the distance function, the inequality (3.7) holds true in an open neighborhood of t^* in $[\hat{t}, \bar{t}]$. It follows that I is open in $[\hat{t}, \bar{t}]$. Henc we conclude that $I = [\hat{t}, \bar{t}]$. Consequently, we have $d(x_0, \hat{x}, \hat{t}) \leq d(x_0, \bar{x}, \bar{t}) + \frac{1}{2}AQ^{-\frac{1}{2}}$.

Claim 2 If (x,t) satisfies (3.3), then the estimate (3.1) holds.

Indeed, consider (x,t) satisfying (3.3). If $(x,t) \in M_{\alpha_n}$, then (3.4) implies that the estimate (3.1) holds. If $(x,t) \notin M_{\alpha_n}$, then we have $|Rm|(x,t) \leq 2Q$ as in the above argument. So (3.1) also holds.

Now we take $\epsilon = \frac{1}{42}$ and $A = 10$. Then $\frac{1}{10}A < 1$ and $(\frac{5}{2}A + 1)\epsilon = 1$. So (3.5) implies

$$B(\bar{x}, Q^{-\frac{1}{2}}, \bar{t}) \subset B(x_0, 1, t) \tag{3.9}$$

for $t \in [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$, and hence

$$\int_{B(\bar{x}, Q^{-\frac{1}{2}}, \bar{t})} |Rm|^{\frac{n}{2}}|_t \leq \delta_0 \tag{3.10}$$

for $t \in [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$. Moreover, Claim 2 implies that the estimate (3.1) holds on $B(\bar{x}, Q^{-\frac{1}{2}}, \bar{t}) \times [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$. We shift $\bar{t}$ to the time origin and rescale g by the factor Q to obtain a Ricci flow solution $\bar{g} = Qg$ on $M \times [-\frac{1}{2}\alpha_n, 0]$. Then we have

$$|Rm|(\bar{x}, 0) = 1, \tag{3.11}$$

and

$$|Rm|(x,t) \le 4 \tag{3.12}$$

whenever

$$-\frac{1}{2}\alpha_n \le t \le 0, d(\bar{x}, x, 0) \le 1.$$

Moreover there holds

$$\int_{B(\bar{x},1,0)} |Rm|^{\frac{n}{2}}|_t \le \delta_0 \tag{3.13}$$

for $t \in [-\frac{1}{2}\alpha_n, 0]$. By the κ-noncollapsedness assumption, we have

$$vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{4}, t)) \ge \frac{\kappa}{4^n}. \tag{3.14}$$

It follows that there is a positive constant $C_1(\kappa, n)$ depending only on κ and n such that

$$C_{S,\bar{g}(t)}(B(\bar{x}, \frac{1}{16}, t)) \le C_1(\kappa, n). \tag{3.15}$$

By the curvature bound (3.12) and the argument in [Ye1] for evolution of the Sobolev constant, we then infer

$$C_{S,2,\bar{g}(t)}(B(\bar{x}, \frac{1}{16}, 0)) \le C_2(\kappa, n) \tag{3.16}$$

for $t \in [-\frac{1}{2}\alpha_n, 0]$, where $C_2(\kappa, n)$ is a positive constant depending only on κ and n. (One can also replace $\frac{1}{16}$ by a smaller radius r_1 such that $B(\bar{x}, r_1, 0) \subset B(\bar{x}, \frac{1}{16}, t)$.)

On the other hand, the curvature bound (3.12) and the Ricci flow equation imply that $B(\bar{x}, \frac{1}{16}, 0) \subset B(\bar{x}, 1, t)$ for $t \in [-\bar{\alpha}_n, 0]$, where $\bar{\alpha}_n \le \frac{1}{2}\alpha_n$ is a positive constant depending only on n. It follows that

$$\int_{B(\bar{x},\frac{1}{16},0)} |Rm|^{\frac{n}{2}}|_t \le \delta_0 \tag{3.17}$$

for $t \in [-\bar{\alpha}_n, 0]$.

Now we appeal to the evolution equation of Rm associated with the Ricci flow

$$\frac{\partial Rm}{\partial t} = \Delta Rm + B(Rm, Rm), \tag{3.18}$$

where B is a certain quadratic form. It implies

$$\frac{\partial}{\partial t}|Rm| \le \Delta|Rm| + c(n)|Rm|^2 \tag{3.19}$$

for a positive constant $c(n)$ depending only on n. On account of (3.12), (3.16) and (3.17) we can apply Theorem 2.1 to (3.19) with $p_0 = \frac{n}{2}$ to deduce

$$
\begin{aligned}
|Rm|(\bar{x},0) &\leq (1+\frac{2}{n})^{\frac{2\sigma_n}{n}} C_2(\kappa,n)^2 C_3(n)\Big(\int_{-\bar{\alpha}_n}^{0}\int_{B(x_0,\frac{1}{16},0)} |Rm|^{\frac{n}{2}}\Big)^{\frac{2}{n}} \\
&\leq (1+\frac{2}{n})^{\frac{2\sigma_n}{n}} C_2(\kappa,n)^2 C_3(n)(\bar{\alpha}_n\delta_0)^{\frac{2}{n}}, \qquad (3.20)
\end{aligned}
$$

where

$$
C_3(n) = \Big(2c(n)n + 2n(n-1) + \frac{n}{2}(1+\frac{n}{2})^2 \cdot \frac{1}{\bar{\alpha}_n} + 64(n+2)^2 e^{4(n-1)\bar{\alpha}_n}\Big)^{\frac{n+2}{n}}.
$$

We deduce $|Rm|(\bar{x},0) \leq \frac{1}{2}$, provided that we define

$$
\delta_0 = \frac{1}{2^{\frac{n}{2}}}(1+\frac{2}{n})^{-\sigma_n} C_2(\kappa,n)^{-n} C_3(n)^{-\frac{n}{2}} \bar{\alpha}_n^{-1}.
$$

But this contradicts (3.11). Hence the estimate (1.4) has been established.

Proof of the estimate (1.5)

Consider a fixed $t_0 \in (0,T)$. If the ratio $\frac{r_0^2}{t_0} \geq 1$, we rescale g by t_0^{-1}. If $\frac{r_0^2}{t_0} \leq 1$, we rescale g by r_0^2. We handle the former case, while the latter is similar. For the rescaled flow $t_0^{-2}g$ on $[0, t_0^{-2}T)$ we have by (1.4)

$$
|Rm|(x,t) \leq 2\alpha_n + \epsilon_0^{-2} \qquad (3.21)
$$

whenever $\frac{1}{2} \leq t < t_0^{-2}T$ and $d(x_0,x,t) < t_0^{-\frac{1}{2}} r_0\epsilon_0$. We rescale the flow further by the factor $\lambda_n \equiv 2\alpha_n + \epsilon_0^{-2}$ to obtain $\bar{g} = \lambda_n t_0^{-2} g$ on $[0, \lambda_n t_0^{-2}T)$. Note that the time t_0 is transformed to $\bar{t}_0 = \lambda_n$. We have for $\bar{g}$

$$
|Rm|(x,t) \leq 1 \qquad (3.22)
$$

whenever $\frac{\lambda_n}{2} \leq t < \lambda_n t_0^{-2}T$ and $d(x_0,x,t) < \sqrt{\lambda_n} t_0^{-\frac{1}{2}} r_0\epsilon_0$. Note $\sqrt{\lambda_n} t_0^{-\frac{1}{2}} r_0\epsilon_0 > 1$. By the κ-noncollapsedness we then deduce

$$
vol_{\bar{g}(t)}(B(x_0,1,t)) \geq \kappa \qquad (3.23)
$$

whenever $\frac{\lambda_n}{2} \leq t < \lambda_n t_0^{-2}T$. Now we can argue as in the above proof of the estimate (1.4) and apply Theorem 2.1 on $B(x_0,\frac{1}{4},\lambda_n) \times [\lambda_n - 1, \lambda_n]$ to derive

$$
|Rm|(x,\lambda_n) \leq C_0(n,\kappa) \sup_{\frac{\lambda_n}{2}-1 \leq t \leq \lambda_n} \Big(\int_{B(x_0,\frac{1}{4},t)} |Rm|^{\frac{n}{2}}(\cdot,t)\, dvol_{\bar{g}(t)}\Big)^{\frac{2}{n}}, \qquad (3.24)
$$

whenever $d(x_0, x, \lambda_n) \leq \frac{1}{8}$. Scaling back to g we then arrive at the desired estimate (1.5) (with t_0 in place of t).

Finally, the smooth convergence result stated in the theorem follows from the estimate (1.4) and an estimate of the distance via the Ricci flow.

■

4 Proof of Theorem B

Proof of Theorem B

Proof of the estimate (1.7)

We present the case of the Ricci flow, while the other two cases are similar.

Assume that the estimate (1.7) fails to hold. Then we carry out the same construction as in the proof of Theorem A-1. We choose again $\epsilon = \frac{1}{42}$ and $A = 10$. Then

$$d(x_0, \bar{x}, \bar{t}) < \frac{1}{2} \tag{4.1}$$

for g. By (1.6) we have for $\bar{g}$

$$\int_{B(\bar{x}, \frac{1}{2}, t)} |Rm|^{\frac{n}{2}}|_t \leq 2^n \delta_0 vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t)) \tag{4.2}$$

for all $t \in (-\frac{1}{2}\alpha_n, 0]$. As before, we also have for $\bar{g}$

$$|Rm|(\bar{x}, 0) = 1 \tag{4.3}$$

and

$$|Rm|(x, t) \leq 4 \tag{4.4}$$

whenever

$$-\frac{1}{2}\alpha_n \leq t \leq 0, d(\bar{x}, x, 0) \leq 1. \tag{4.5}$$

By [Theorem 4.1, An] and (4.4) we have

$$C_{S,2,\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t)) \leq \frac{C_5(n)}{vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t))^{\frac{1}{n}}} \tag{4.6}$$

for $t \in [-\frac{1}{2}\alpha_n, 0]$, with a positive constant $C_5(n)$ depending only on n. On the other hand, (4.4) implies

$$B(\bar{x}, \frac{1}{4}, t_1) \subset B(\bar{x}, \frac{1}{3}, t_2) \subset B(\bar{x}, \frac{1}{2}, t_3) \tag{4.7}$$

for all $t_1, t_2, t_3 \in [-\bar{\alpha}_n, 0]$, with a positive constant $\bar{\alpha}_n \leq \frac{1}{2}\alpha_n$ depending only on n. Consequently, we have

$$\int_{B(\bar{x}, \frac{1}{4}, 0)} |Rm|^{\frac{n}{2}}|_t \leq 3^n \delta_0 vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{3}, t)) \tag{4.8}$$

and

$$C_{S,2,\bar{g}(t)}(B(\bar{x}, \frac{1}{4}, 0)) \leq \frac{C_5(n)}{vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t)))^{\frac{1}{n}}} \tag{4.9}$$

for all $t \in [-\bar{\alpha}_n, 0]$. Moreover, (4.7) combined with (4.4) leads via the Ricci flow equation to

$$\min_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t)) \geq e^{-4n(n-1)\bar{\alpha}_n} \max_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{3}, t)) \tag{4.10}$$

for each $t \in [-\bar{\alpha}_n, 0]$. Now we apply Theorem 2.1 to deduce

$$\begin{aligned} |Rm|(\bar{x}, 0) &\leq (1 + \frac{2}{n})^{\frac{2\sigma_n}{n}} \frac{C_5(n)^2}{\min\limits_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t)))^{\frac{2}{n}}} C_6(n) \Big(\int_{-\bar{\alpha}_n}^0 \int_{B(\bar{x}, \frac{1}{4}, 0)} |Rm|^{\frac{n}{2}} \Big)^{\frac{2}{n}} \\ &\leq \frac{9(1 + \frac{2}{n})^{\frac{2\sigma_n}{n}} C_5(n)^2 C_6(n) (\delta_0 \bar{\alpha}_n)^{\frac{n}{2}}}{\min\limits_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t)))^{\frac{2}{n}}} \max_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{3}, t))^{\frac{2}{n}} \\ &\leq 9(1 + \frac{2}{n})^{\frac{2\sigma_n}{n}} C_5(n)^2 C_6(n) (\bar{\alpha}_n \delta_0)^{\frac{2}{n}} e^{8(n-1)\bar{\alpha}_n}, \end{aligned} \tag{4.11}$$

with a suitable positive constant $C_6(n)$ depending only on n. Choosing

$$\delta_n = \frac{1}{3^n}(1 + \frac{2}{n})^{-\sigma_n} C_5(n)^{-n} C_6(n)^{-\frac{n}{2}} \bar{\alpha}_n^{-1} e^{-8(n-1)\bar{\alpha}_n}$$

we then obtain $|Rm|(\bar{x}, 0) \leq \frac{1}{2}$, contradicting (4.3).

Proof of the estimate (1.8)

Consider a fixed $t_0 \in (0, T)$. As in the corresponding part of the proof of Theorem A, we present the case $\frac{r_0^2}{t_0} \geq 1$, and rescale g by t_0^{-1}. Again we have for the rescaled flow $t_0^{-2} g$ on $[0, t_0^{-2} T)$

$$|Rm|(x, t) \leq 2\alpha_n + \epsilon_0^{-2} \tag{4.12}$$

whenever $\frac{1}{2} \le t < t_0^{-2}T$ and $d(x_0, x, t) < t_0^{-\frac{1}{2}} r_0 \epsilon_0$. As before, we rescale the flow further by the factor $\lambda_n \equiv 2\alpha_n + \epsilon_0^{-2}$ to obtain $\bar{g} = \lambda_n t_0^{-2} g$ on $[0, \lambda_n t_0^{-2}T)$. Again, the time t_0 is transformed to $\bar{t}_0 = \lambda_n$. We have for $\bar{g}$

$$|Rm|(x, t) \le 1 \tag{4.13}$$

whenever $\frac{\lambda_n}{2} \le t < \lambda_n t_0^{-2}T$ and $d(x_0, x, t) \le \sqrt{\lambda_n} t_0^{-\frac{1}{2}} r_0 \epsilon_0$. Since $\sqrt{\lambda_n} t_0^{-\frac{1}{2}} r_0 \epsilon_0 > 1$, we can argue in the same way as in the above proof of (1.7), using the radius $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$. Note that the time λ_n corresponds to 0 there. We deduce

$$|Rm|(x, \lambda_n) \le \bar{C}_0(n) \sup_{\frac{\lambda_n}{2} - 1 \le t \le \lambda_n} \left(\frac{\int_{B(x_0, \frac{1}{4}, t)} |Rm|^{\frac{n}{2}}(\cdot, t)}{vol_{\bar{g}}(B(x_0, \frac{1}{4}, t))} \right)^{\frac{2}{n}}, \tag{4.14}$$

with a positive constant $\bar{C}_0(n)$ depending only on n, whenever $d(x_0, x, \lambda_n) \le \frac{1}{8}$. Scaling back to g we then arrive at the desired estimate (1.5) (with t_0 in place of t).

5 Proof of Theorem C

Proof of Theorem C

We establish the condition (1.6). Then the theorem follows from Theorem B. By rescaling we can assume $r_0 = 1$. Then (1.9) becomes

$$Ric \ge -(n-1)g. \tag{5.1}$$

By (1.10), we have now

$$\int_{B(x,r,1)} |Rm|^{\frac{n}{2}} dvol_{g(t)} \le \delta_0 vol_{g(t)}(B(x, r, 1)) \tag{5.2}$$

for all $t \in [0, T]$. By Bishop-Gromov relative volume comparison, we have

$$vol_{g(t)}(B(x, R, t)) \le \frac{v_{-1}(R)}{v_{-1}(r)} vol_{g(t)}(B(x, r, t)) \le C(n) \frac{vol_{g(t)}(B(x, r, t))}{r^n}, \tag{5.3}$$

with a positive constant $C(n)$ depending only on n, provided that $t \in [0, T], d(x_0, x, t) < 1$, and $0 < r < R \le 1 - d(x_0, x, t)$. Here $v_{-1}(r)$ denotes the volume of a geodesic ball of radius r in $\mathbf{H}^n$, the n-dimensional hyperbolic space (of sectional curvature -1). If $t \in [0, T], d(x_0, x, t) \le \frac{1}{4}$, we then have $B(x_0, \frac{1}{4}, t) \subset B(x, \frac{1}{2}, t) \subset B(x_0, 1, t)$. Consequently,

$$\begin{aligned} vol_{g(t)}(B(x, r, t)) &\ge C(n)^{-1} r^n vol_{g(t)}(B(x, \frac{1}{2}, t)) \ge C(n)^{-1} r^n vol_{g(t)}(B(x_0, \frac{1}{4}, t)) \\ &\ge 4^{-n} C(n)^{-2} r^n vol_{g(t)}(B(x_0, 1, t)) \end{aligned} \tag{5.4}$$

for $0 < r \leq \frac{1}{2}$. Hence we infer

$$\begin{aligned}\int_{B(x,r,t)} |Rm|^{\frac{n}{2}} dvol_{g(t)} &\leq \int_{B(x_0,1,t)} |Rm|^{\frac{n}{2}} dvol_{g(t)} \leq \delta_0 vol_{g(t)}(B(x_0,1,t)) \\ &\leq 4^n C(n)^2 \delta_0 \frac{vol_{g(t)}(B(x,r,t))}{r^n} \qquad (5.5)\end{aligned}$$

whenever $t \in [0,T], d(x_0,x,t) \leq \frac{1}{4}r_0$ and $0 < r < \frac{1}{2}r_0$. Choosing δ_0 to be the δ_0 in Theorem B-1 multiplied by $4^{-n}C(n)^{-2}$ and replacing r_0 by $\frac{r_0}{2}$ we then have all the conditions of Theorem B-1. The desired estimate follows. By the proof of Theorem B-1, (1.7) actually holds with $\epsilon_0 = \frac{1}{42}$. Hence we obtain $\epsilon_0 = \frac{1}{84}$ now.

■

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