Curvature Estimates for the Ricci Flow II

Rugang Ye
Department of Mathematics
University of California, Santa Barbara

1 Introduction

In this paper we present several curvature estimates and convergence results for solutions of the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric \tag{1.1}$$

and its normalized versions, such as the volume normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric + \frac{2}{n}\mathcal{S}g \tag{1.2}$$

with $\mathcal{S}$ denoting the total scalar curvature and n denoting the dimension of the manifold, and the λ-normalized Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric + \lambda g \tag{1.3}$$

with a constant λ. (Since Einstein metrics are stationary solutions of the volume normalized Ricci flow, they are included as a special case.) The said curvature estimates are space-time analogues of the curvaure estimates in [Ye3], and depend on the smallness of certain local space-time $L^{\frac{n+2}{2}}$ integrals of the norm of the Riemann curvature tensor. On the other hand, the said convergence results require finiteness of space-time $L^{\frac{n+2}{2}}$ integrals of the norm of the Riemann curvature tensor. Note that these curvature estimates and convergence results also serve as characterizations of blow-up singularities (see e.g. Remark 2 below).

To formulate our results, we need some terminologies. Consider a connected Riemannian manifold (M, g) (g denotes the metric) possibly with boundary, and $x \in M$. If x is in the interior of M, we define the distance $d(x, \partial M)$ to be $\sup\{r > 0 : B(x, r)$ is compact and contained in the interior of $M\}$, where $B(x, r)$ denotes the closed geodesic ball of center x and radius r. If M has a boundary and $x \in \partial M$, then $d(x, \partial M)$ is the ordinary distance from x to ∂M and equals zero. (For example, $d(x, \partial M) = \infty$ if M is closed.)

Let $g = g(t)$ be a family of metrics on M. Then $d(x, y, t)$ denotes the distance between $x, y \in M$ with respect to the metric $g(t)$, and $B(x, r, t)$ denotes the closed geodesic ball of center $x \in M$ and radius r with respect to the metric $g(t)$.

We set $\alpha_n = \frac{1}{40(n-1)}$, $\epsilon_0 = \frac{1}{168}$ and $\epsilon_1 = \frac{\epsilon_0}{8\sqrt{1+2\alpha_n\epsilon_0^2}}$. (These constants are not meant to be optimal. They can be improved by closely examining the proofs.)

We divide our results into several types. In each type, the first theorem is a local curvature estimate, the second theorem a global convergence result, and the third theorem a local convergence result.

Type A

Results of this type involve straight (i.e. non-weighted) space-time $L^{\frac{n+2}{2}}$-integals of the norm of the Riemann curvature tensor. Theorem A-2 (the convergence result) does not involve any additional quantity or condition. Theorem A-1 and Theorem A-2 involve the condition of κ-noncollapsedness. Note that By [Theorem 4.1, P], a smooth solution of the Ricci flow on $M \times [0, T)$ for a closed manifold M and a finite T is κ-noncollapsed on the scale $\sqrt{T}$, where κ depends on the initial metric and T.

Theorem A-1 *For each positive number κ and each natural number $n \geq 3$ there are positive constants $\delta_0 = \delta_0(n, \kappa)$ and $C_0 = C_0(n, \kappa)$ depending only on n and κ with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow or the volume normalized Ricci flow on $M \times [0, T)$ for a connected manifold M of dimension $n \geq 3$ and some (finite or infinite) $T > 0$, which is κ-noncollapsed on the scale of ρ for some $\kappa > 0$ and $\rho > 0$. Consider $x_0 \in M$ and $0 < r_0 \leq \rho$,which satisfy $r_0 \leq diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T)$. Assume that*

$$\int_0^T \int_{B(x_0, r_0, t)} |Rm|^{\frac{n+2}{2}}(\cdot, t) dvol_{g(t)} dt \leq \delta_0. \tag{1.4}$$

Then we have

$$|Rm|(x, t) \leq \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2} \tag{1.5}$$

whenever $t \in (0, T)$ and $d(x_0, x, t) < \epsilon_0 r_0$, and

$$|Rm|(x, t) \leq C_0 \max\{r_0^{-2}, t^{-1}\} (\int_0^T \int_{B(x_0, r_0, t)} |Rm|^{\frac{n+2}{2}}(\cdot, t) dvol_{g(t)} dt)^{\frac{2}{n+2}} \tag{1.6}$$

whenever $0 < t < T$ and $d(x_0, x, t) \leq \epsilon_1 \min\{r_0, \sqrt{t}\}$. (Obviously, the estimates (1.5) and (1.6) hold on $[0, T]$ provided that T is finite and the assumptions hold on $[0, T]$. This remark also applies to the results below.)

The same result holds for a solution g of the λ-normalized Ricci flow, with δ_0 also depending on $|\lambda|$.

Theorem A-2 *Let $g = g(t)$ be a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow on $M \times [0, T)$ for an n-dimensional manifold M and some finite $T > 0$, such that $g(t)$ is complete for each $t \in [0, T)$. Assume*

$$\int_0^T \int_M |Rm|^{\frac{n+2}{2}} dvol_{g(t)} dt < \infty. \tag{1.7}$$

Then $g(t)$ converges smoothly to a smooth metric on M as $t \to T$. Consequently, $g(t)$ extends to a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow over $[0, T']$ for some $T' > T$. Here and below, smooth convergence means smooth convergence on each compact subset. .

Remark 1 This result is optimal in the sense that if we replace $\frac{n+2}{2}$ by a smaller exponent, then the conclusion fails to hold. This is demonstrated by the example of the evolving sphere. This remark also applies to the results below.

Theorem A-3 *Let $g = g(t)$ be a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow on $M \times [0, T)$ for a connected manifold M of dimension $n \geq 3$ and some (finite or infinite) $T > 0$,which is κ-noncollapsed on the scale of ρ for some $\kappa > 0$ and $\rho > 0$. Consider $x_0 \in M$ and $0 < r_0 \leq \rho$, which satisfy $r_0 \leq diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T)$. Assume that*

$$\int_0^T \int_{B(x_0, r_0, t)} |Rm|^{\frac{n+2}{2}} (\cdot, t) dvol_{g(t)} dt < \infty. \tag{1.8}$$

If T is finite, then $g(t)$ converges smoothly to a smooth metric $g(T)$ on the direct limit $\underline{\lim}_{t \to T} \mathring{B}(x_0, \epsilon_0, t)$. Moreover, $\mathring{B}(x_0, \epsilon_0 r_0, T) = \underline{\lim}_{t \to T} \mathring{B}(x_0, \epsilon_0 r_0, t)$. If $T = \infty$, we have similar smooth subconvergence of $g(t)$ as $t \to \infty$.

Remark 2 Theorem A-1 and Theorem A-2 can be rephrased as follows: A (global or local) solution of the Ricci flow (the volume normalized Ricci low, or the λ-normalized Ricci flow) blows up at T, if and only if the space-time integral of $|Rm|^{\frac{n+2}{2}}$ up to T is infinite. This can be used to analyse blow-ups of the Ricci flow. For example, careful rescalings produce blow-up limits with the special feature of infinite space-time integral of $|Rm|^{\frac{n+2}{2}}$. This will be discussed in more details elsewhere. This remark also applies to the results below.

Type B

Results of this type do not use the condition of κ-noncollapsedness. Instead, they involve space-time $L^{\frac{n+2}{2}}$ integrals of the norm of Rm over balls of varying center and radius measured against a volume ratio.

Theorem B-1 *For each natural number $n \geq 3$ there is a positive constant $\delta_0 = \delta_0(n)$ depending only on n with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow or the volume normalized Ricci flow on $M \times [0,T)$ for a connected manifold M of dimension $n \geq 3$ and some (finite or infinite) $T > 0$. Consider $x_0 \in M$ and $r_0 > 0$,which satisfy $r_0 \leq diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0,T)$. I. Assume that*

$$\int_0^T \frac{r^n}{vol_{g(t)}(B(x,r,t))^*} \int\limits_{B^*(x,r,t)} |Rm|^{\frac{n+2}{2}}(\cdot,t) dvol_{g(t)} dt \leq \delta_0, \tag{1.9}$$

for all $0 < r \leq \frac{r_0}{2}$ and $x \in M$, where $B^*(x,r,t) = B(x,r,t)$, $vol_{g(t)}(B(x,r,t))^* = vol_{g(t)}(B(x,r,t))$ if $x \in B(x_0, \frac{r_0}{2}, t)$, and $B^*(x,r,t) = \{x\}$, $vol_{g(t)}(B(x,r,t))^* = r^n$ if $x \notin B(x_0, \frac{r_0}{2}, t)$. (Thus, if $x \notin B(x_0, \frac{r_0}{2}, t)$, then the value of the integrand for the time integral is zero at t.) Then we have

$$|Rm|(x,t) \leq \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2} \tag{1.10}$$

whenever $t \in (0,T)$ and $d(x_0, x, t) < \epsilon_0 r_0$, and

$$|Rm|(x,t) \leq C_0 r_1^{-\frac{4}{n+2}} \left(\int_0^T \frac{\int\limits_{B(x_0,2r_1,t)} |Rm|^{\frac{n+2}{2}}(\cdot,t) dvol_{g(t)}}{vol_{g(t)}(B(x_0,2r_1,t))} dt \right)^{\frac{2}{n+2}} \tag{1.11}$$

whenever $0 < t < T$ and $d(x_0, x, t) \leq r_1$, where $r_1 = \epsilon_1 \min\{r_0, \sqrt{t}\}$.

The same results hold for the λ-normalized Ricci flow, with δ_0 also depending on $|\lambda|$.

Theorem B-2 *Let $g = g(t)$ be a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow on $M \times [0,T)$ for an n-dimensional manifold M and some (finite or infinite) $T > 0$, such that $g(t)$ is complete for each $t \in [0,T)$. Assume*

$$\int_0^T \sup_{x \in M, 0 < r \leq \frac{r_0}{2}} \frac{r^n}{vol_{g(t)}(B(x,r,t))} \int_{B(x,r,t)} |Rm|^{\frac{n+2}{2}}(\cdot,t) dvol_{g(t)} < \infty \tag{1.12}$$

for some $r_0 > 0$. If T is finite, then $g(t)$ converges smoothly to a smooth metric on M as $t \to T$. Consequently, $g(t)$ extends to a smooth solution of the Ricci flow, *the volume normalized Ricci flow or the λ-normalized Ricci flow over $[0, T']$ for some $T' > T$. If T is infinite, then $g(t)$ subconverges smoothly as $t \to T$.*

Remark 3 Note that there is no assumption on the relation between r_0 and the diameter. Hence no control over the diameter is assumed. This remark also applies to Theorem C-3 below.

Theorem B-3 *For each natural number $n \geq 3$ there is a positive constant $\delta_0 = \delta_0(n)$ depending only on n with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow, the volume normalized Ricci flow or the λ-normalized Ricci flow on $M \times [0, T)$ for a connected manifold M of dimension $n \geq 3$ and some (finite or infinite) $T > 0$. Consider $x_0 \in M$ and $r_0 > 0$,which satisfy $r_0 \leq diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T)$. Assume*

$$\int_0^T \left(\sup_{0<r\leq \frac{r_0}{2}, x\in B(x_0, \frac{r}{2}, t)} \frac{r^n}{vol_{g(t)}(B(x,r,t))} \int_{B(x,r,t)} |Rm|^{\frac{n+2}{2}}(\cdot, t) dvol_{g(t)} \right) dt < \infty. \tag{1.13}$$

If T is finite, then $g(t)$ converges smoothly to a smooth metric $g(T)$ on the direct limit $\underline{\lim}_{t\to T} \mathring{B}(x_0, \epsilon_0 r_0, t)$. Moreover, $\mathring{B}(x_0, \epsilon_0 r_0, T) = \underline{\lim}_{t\to T} \mathring{B}(x_0, \epsilon_0, t)$. If $T = \infty$, then we have similar smooth subconvergence of $g(t)$ as $t \to T$.

Type C

Results of this type do not use the condition of κ-noncollapsedness, and involve only a fixed center and a fixed radius for space-time $L^{\frac{n+2}{2}}$ integrals of the norm of the Riemann curvature tensor. But a lower bound for the Ricci curvature is assumed.

Theorem C-1 *For each natural number $n \geq 3$ there is a positive constant $\delta_0 = \delta_0(n)$ depending only on n with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow or the volume normalized Ricci flow on $M \times [0, T)$ for a connected manifold M of dimension $n \geq 3$ and some (finite or infinite) $T > 0$. Consider $x_0 \in M$ and $r > 0$, which satisfy $r_0 \leq diam_{g(t)}(M)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T)$. Assume that*

$$Ric(x, t) \geq -\frac{n-1}{r_0^2} g(x, t) \tag{1.14}$$

whenever $t \in [0, T)$ and $d(x_0, x, t) \leq r_0$, and that

$$\int_0^T \frac{r_0^n}{vol_{g(t)}(B(x_0, r_0, t))} \int_{B(x_0,r_0,t)} |Rm|^{\frac{n+2}{2}}(\cdot, t) dvol_{g(t)} dt \leq \delta_0. \tag{1.15}$$

Then we have

$$|Rm|(x,t) \leq \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2} \tag{1.16}$$

whenever $t \in (0,T)$ *and* $d(x_0, x, t) < \epsilon_0 r_0$, *and*

$$|Rm|(x,t) \leq C_0 r_1^{-\frac{4}{n+2}} \left(\int_0^T \frac{\int_{B(x_0,2r_1,t)} |Rm|^{\frac{n+2}{2}}(\cdot,t) dvol_{g(t)}}{vol_{g(t)}(B(x_0,2r_1,t))} dt \right)^{\frac{2}{n+2}} \tag{1.17}$$

whenever $0 < t < T$ *and* $d(x_0, x, t) \leq r_1$, *where* $r_1 = \epsilon_1 \min\{r_0, \sqrt{t}\}$.

The same results hold for the λ*-normalized Ricci flow. with* δ_0 *also depending on* $|\lambda|$.

Theorem C-2 *Let* $g = g(t)$ *be a smooth solution of the Ricci flow, the volume normalized Ricci flow, or the* λ*-normalized Ricci flow on* $M \times [0,T)$ *for an n-dimensional manifold* M *and some (finite or infinite)* $T > 0$, *such that* $g(t)$ *is complete for each* $t \in [0,T)$. *Assume that (1.14) holds for all* $x \in M$ *and* $t \in [0,T)$, *and*

$$\int_0^T \sup_{x \in M} \frac{1}{vol_{g(t)}(B(x,r_0,t))} \int_{B(x,r_0,t)} |Rm|^{\frac{n+2}{2}}(\cdot,t) dvol_{g(t)} dt < \infty \tag{1.18}$$

for some $r_0 > 0$. *If* $T < \infty$, *then* $g(t)$ *converges smoothly to a smooth metric on* M *as* $t \to T$. *Consequently,* $g(t)$ *extends to a smooth solution of the Ricci flow, the volume normalized Ricci flow or the* λ*-normalized Ricci flow over* $[0,T']$ *for some* $T' > T$. *If* $T = \infty$, *then* $g(t)$ *subconverges smoothly as* $t \to T$.

Theorem C-3 *For each natural number* $n \geq 3$ *there is a positive constant* $\delta_0 = \delta_0(n)$ *depending only on* n *with the following property. Let* $g = g(t)$ *be a smooth solution of the Ricci flow,the volume normalized Ricci flow or the* λ*-normalized Ricci flow on* $M \times [0,T)$ *for a connected manifold* M *of dimension* $n \geq 3$ *and some (finite or infinite)* $T > 0$. *Consider* $x_0 \in M$ *and* $r_0 > 0$, *which satisfy* $r_0 \leq diam_{g(t)}(M)$ *and* $r_0 \leq d_{g(t)}(x_0, \partial M)$ *for each* $t \in [0,T)$. *Assume that (1.14) holds whenever* $0 \leq t < T$ *and* $d(x_0, x, t) < r_0$, *and that*

$$\int_0^T \frac{1}{vol_{g(t)}(B(x_0,r_0,t))} \int_{B(x_0,r_0,t)} |Rm|^{\frac{n+2}{2}}(\cdot,t) dvol_{g(t)} dt < \infty. \tag{1.19}$$

If $T < \infty$, *then* $g(t)$ *converges smoothly to a smooth metric* $g(T)$ *on the direct limit* $\underline{\lim}_{t \to T} \mathring{B}(x_0, \epsilon_0 r_0, t)$. *Moreover,* $\mathring{B}(x_0, \epsilon_0 r_0, T) = \underline{\lim}_{t \to T} \mathring{B}(x_0, \epsilon_0, t)$. *If* $T = \infty$, *we have similar smooth subconvergence of* $g(t)$ *as* $t \to T$.

Similar results hold for many other evolution equations. This will be presented elsewhere.

The curvature estimates in this paper were obtained some time ago.

Analogous results involving other types of L^p integrals of $|Rm|$ (including $p < \frac{n}{2}$) will be presented in a sequel of this paper.

2 Proofs of Theorems of Type A

Proof of Theorem A-1

Proof of the estimate (1.5)

We handle the case of the Ricci flow. The other two cases are similar. (For the volume normalized Ricci flow, we can rescale to achieve volume 1.) The proof is similar to [Proof of Theorem A, Ye3]. To make the proof clear, we'll repeat some arguments in [Ye3]. By rescaling, we can assume $r_0 = 1$. Assume that the estimate (1.5) does not hold. Then we can find for each $\epsilon > 0$ a Ricci flow solution $g = g(t)$ on $M \times [0, T)$ for some M and $T > 0$ with the properties as postulated in the statement of the theorem, such that $|Rm|(x, t) > \alpha_n t^{-1} + \epsilon^{-2}$ for some $(x, t) \in M \times (0, T)$ satisfying $d(x_0, x, t) < \epsilon$.

We denote by M_{α_n} the set of pairs (x, t) such that $|Rm|(x, t) \geq \alpha_n t^{-1}$. For an arbitary positive number $A > 1$ such that $(2A + 1)\epsilon \leq \frac{1}{2}$, we choose as in [Proof of Theorem 10.1, P] and [Proof of Theorem A, Ye3] a point $(\bar{x}, \bar{t}) \in M_{\alpha_n}$ with $0 < \bar{t} \leq \epsilon^2, d(x_0, \bar{x}, \bar{t}) < (2A + 1)\epsilon$, such that $|Rm|(\bar{x}, \bar{t}) > \alpha_n \bar{t}^{-1} + \epsilon^{-2}$ and

$$|Rm|(x, t) \leq 4|Rm|(\bar{x}, \bar{t}) \tag{2.1}$$

whenever

$$(x, t) \in M_{\alpha_n}, 0 < t \leq \bar{t}, d(x_0, x, t) \leq d(x_0, \bar{x}, \bar{t}) + A|Rm|(\bar{x}, \bar{t})^{-\frac{1}{2}}. \tag{2.2}$$

We set $Q = |Rm|(\bar{x}, \bar{t})$. By [Proof of Theorem A, Ye3], the following two claims hold.

Claim 1 If

$$\bar{t} - \frac{1}{2}\alpha_n Q^{-1} \leq t \leq \bar{t}, d(\bar{x}, x, \bar{t}) \leq \frac{1}{10}AQ^{-\frac{1}{2}}, \tag{2.3}$$

then

$$d(x_0, x, t) \leq d(x_0, \bar{x}, \bar{t}) + \frac{1}{2}AQ^{-\frac{1}{2}}. \tag{2.4}$$

Claim 2 If (x, t) satisfies (2.3), then the estimate (2.1) holds.

Note that (2.4) implies

$$d(x_0, x, t) \leq (2A+1)\epsilon + \frac{1}{2}AQ^{-\frac{1}{2}} \leq (\frac{5}{2}A+1)\epsilon \tag{2.5}$$

for (x, t) satisfying (2.3).

Now we take $\epsilon = \frac{1}{42}$ and $A = 10$. Then $\frac{1}{10}A < 1$ and $(\frac{5}{2}A+1)\epsilon = 1$. So (2.5) implies

$$B(\bar{x}, Q^{-\frac{1}{2}}, \bar{t}) \subset B(x_0, 1, t) \tag{2.6}$$

for $t \in [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$, and hence the condition (1.4) leads to

$$\int_{\bar{t}-\frac{1}{2}\alpha_n Q^{-1}}^{\bar{t}} \int_{B(\bar{x}, Q^{-\frac{1}{2}}, \bar{t})} |Rm|^{\frac{n+2}{2}} \leq \delta_0. \tag{2.7}$$

Moreover, Claim 2 implies that the estimate (2.1) holds on $B(\bar{x}, Q^{-\frac{1}{2}}, \bar{t}) \times [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$. We shift $\bar{t}$ to the time origin and rescale g by the factor Q to obtain a Ricci flow solution $\bar{g} = Qg$ on $M \times [-\frac{1}{2}\alpha_n, 0]$. Then we have

$$|Rm|(\bar{x}, 0) = 1, \tag{2.8}$$

and

$$|Rm|(x, t) \leq 4 \tag{2.9}$$

whenever

$$-\frac{1}{2}\alpha_n \leq t \leq 0, d(\bar{x}, x, 0) \leq 1.$$

Moreover there holds

$$\int_{-\frac{1}{2}\alpha_n}^{0} \int_{B(\bar{x}, 1, 0)} |Rm|^{\frac{n}{2}} \leq \delta_0. \tag{2.10}$$

By the κ-noncollapsedness assumption, we have

$$vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{4}, t)) \geq \frac{\kappa}{4^n}. \tag{2.11}$$

It follows that there is a positive constant $C_1(\kappa, n)$ depending only on κ and n such that

$$C_{S,\bar{g}(t)}(B(\bar{x}, \frac{1}{16}, t)) \leq C_1(\kappa, n). \tag{2.12}$$

By the curvature bound (2.9) and the argument in [Ye3] for evolution of the Sobolev constant, we then infer

$$C_{S,2,\bar{g}(t)}(B(\bar{x},\frac{1}{16},0)) \le C_2(\kappa,n) \tag{2.13}$$

for $t \in [-\frac{1}{2}\alpha_n, 0]$, where $C_2(\kappa,n)$ is a positive constant depending only on κ and n. (One can also replace $\frac{1}{16}$ by a smaller radius r_1 such that $B(\bar{x},r_1,0) \subset B(\bar{x},\frac{1}{16},t)$.)

On the other hand, the curvature bound (2.9) and the Ricci flow equation imply that $B(\bar{x},\frac{1}{16},0) \subset B(\bar{x},1,t)$ for $t \in [-\bar{\alpha}_n, 0]$, where $\bar{\alpha}_n \le \frac{1}{2}\alpha_n$ is a positive constant depending only on n. It follows that

$$\int_{-\bar{\alpha}_n}^{0} \int_{B(\bar{x},\frac{1}{16},0)} |Rm|^{\frac{n+2}{2}} \le \delta_0. \tag{2.14}$$

As in [Ye3], we now appeal to the evolution equation of Rm associated with the Ricci flow

$$\frac{\partial Rm}{\partial t} = \Delta Rm + B(Rm, Rm), \tag{2.15}$$

where B is a certain quadratic form. It implies

$$\frac{\partial}{\partial t}|Rm| \le \Delta |Rm| + c(n)|Rm|^2 \tag{2.16}$$

for a positive constant $c(n)$ depending only on n. On account of (2.9), (2.13) and (2.14) we can apply [Theorem 2.1, Ye3] to (2.16) with $p_0 = \frac{n+2}{2}$ to deduce

$$\begin{aligned} |Rm|(\bar{x},0) &\le (1+\frac{2}{n})^{\frac{2\sigma_n}{n+2}} C_2(\kappa,n)^{\frac{2n}{n+2}} C_3(n) \Big(\int_{-\bar{\alpha}_n}^{0} \int_{B(x_0,\frac{1}{16},0)} |Rm|^{\frac{n+2}{2}} \Big)^{\frac{2}{n+2}} \\ &\le (1+\frac{2}{n})^{\frac{2\sigma_n}{n+2}} C_2(\kappa,n)^{\frac{2n}{n+2}} C_3(n) \delta_0^{\frac{2}{n+2}}, \end{aligned} \tag{2.17}$$

where

$$C_3(n) = 2c(n)(n+2) + 2n(n-1) + \frac{n(n+2)^2}{8} \cdot \frac{1}{\bar{\alpha}_n} + 64(n+2)^2 e^{4(n-1)\bar{\alpha}_n}.$$

We deduce $|Rm|(\bar{x},0) \le \frac{1}{2}$, provided that we define

$$\delta_0 = 2^{-\frac{n+2}{2}}(1+\frac{2}{n})^{-\sigma_n} C_2(\kappa,n)^{-n} C_3(n)^{-\frac{n+2}{2}}.$$

But this contradicts (2.8). Hence the estimate (1.5) has been proven.

Proof of the estimate (1.6)

This is similar to the corresponding part in [Proof of Theorem A, Ye3].

■

Lemma 2.1 *Set $V(t) = vol_{g(t)}(M)$ and $I(t) = \int_0^t \int_M |Rm|^{\frac{n}{2}+1}$. Then*

$$V(t) \leq V_*, \tag{2.18}$$

where

$$V_* = V(0) + (c(n)I(T))^{\frac{2}{n+2}} \left[V(0) + \left(1 + (c(n)I(T))^{\frac{2}{n+2}}\right) \frac{2t}{n+2}\right]^{\frac{n}{n+2}},$$

with $c(n)$ denoting a positive constant depending only on n.

Proof. Let $V = V(t)$ denote the volume of $g(t)$. Then we have

$$\frac{dV}{dt} = -\int_M R. \tag{2.19}$$

Hence

$$\begin{aligned} V(t) - V(0) &= -\int_0^t \int_M R \leq (\int_0^T \int_M |R|^{\frac{n}{2}+1})^{\frac{2}{n+2}} (\int_0^t V(s)ds)^{\frac{n}{n+2}} \\ &\leq (c(n)I(T))^{\frac{2}{n+2}} (\int_0^t V(s)ds)^{\frac{n}{n+2}} \end{aligned} \tag{2.20}$$

with a positive constant $c(n)$ depending only on n. We set $\phi(t) = \int_0^t V(s)ds$. Then

$$\phi'(t) \leq V(0) + (c(n)I(T))^{\frac{2}{n+2}} \phi(t)^{\frac{n}{n+2}}. \tag{2.21}$$

We infer for $\psi = \phi + V(0)$

$$\psi' \leq V(0) + (c(n)I(T))^{\frac{2}{n+2}} \psi^{\frac{n}{n+2}} \tag{2.22}$$

and hence

$$\frac{\psi'}{\psi^{\frac{n}{n+2}}} \leq 1 + (c(n)I(T))^{\frac{2}{n+2}}. \tag{2.23}$$

It follows that

$$\phi^{\frac{2}{n+2}} \leq \psi^{\frac{2}{n+2}} \leq V(0) + (1 + (c(n)I(T))^{\frac{2}{n+2}}) \frac{2t}{n+2}. \tag{2.24}$$

Then

$$V(t) \leq V(0) + (c(n)I(T))^{\frac{2}{n+2}} \left[V(0) + \left(1 + (c(n)I(T))^{\frac{2}{n+2}}\right) \frac{2t}{n+2}\right]^{\frac{n}{n+2}}. \tag{2.25}$$

■

Proof of Theorem A-2

We handle the case of the Ricci flow. The other two cases are similar. By [Theorem 4.1, P], g is κ-noncollapsed on the scale $\sqrt{T}$ for some κ depending on n, T and $g(0)$.

Since the integral $\int_0^T \int_M |Rm|^{\frac{n+2}{2}}$ is finite, we have

$$\int_{T_0}^T \int_M |Rm|^{\frac{n+2}{2}} \leq \delta_0 \tag{2.26}$$

for some $0 < T_0 < T$. By Theorem A-1, we have

$$|Rm|(x,t) \leq \alpha_n \frac{2}{T - T_0} + \epsilon_0^{-2} diam_{g(t)}(M)^{-2}, \tag{2.27}$$

for $\frac{T_0+T}{2} \leq t < T$ satisfying $diam_{g(t)} = \inf_{T_0 \leq t' \leq t} diam_{g(t')}(M)$.

Claim 1 $\liminf_{t \to T} diam_{g(t)}(M) > 0$.

Proof of Claim 1

Obviously, we only need to handle the case that M is closed. We rescale g to normalize volume. Thus we set

$$\lambda(t) = e^{\frac{2}{n}\int_0^t \int_M R},$$

$\tau(t) = \int_0^t \lambda(s)ds$ and

$$\tilde{g}(\tau) = \lambda(t(\tau))g(t(\tau)),$$

where $t(\tau)$ is the inverse of $\tau(t)$. Then $\bar{g}(0) = g(0)$ and $\bar{g}(\tau)$ satisfies the volume normalized Ricci flow. Note that the time interval $[0, T)$ is converted into the interval $[0, \Lambda)$ under the function $\tau(t)$, where $\Lambda = \int_0^T \lambda(t)dt$.

We have

$$\int_0^T \int_M |R| \leq (\int_0^T \int_M |R|^{\frac{n+2}{2}})^{\frac{2}{n+2}} (\int_0^T V(t))^{\frac{n}{n+2}} < \infty \tag{2.28}$$

on account of Lemma 2.1. Hence $\lambda(t)$ is bounded away from zero and bounded from above on $[0, T)$. Let $\Lambda_0 = \tau(T_0)$. We have

$$\begin{aligned} \int_{\Lambda_0}^\Lambda \int_M |Rm|^{\frac{n+2}{2}}|_{\bar{g}} dvol_{\bar{g}} d\tau &= \int_{\Lambda_0}^\Lambda \int_M (|Rm|^{\frac{n+2}{2}}|_g)(\cdot, t(\tau))\lambda^{-1}(t(\tau)) dvol_g(t(\tau))d\tau \\ &= \int_{T_0}^T |Rm|^{\frac{n+2}{2}}|_g dvol_g dt \leq \delta_0. \end{aligned} \tag{2.29}$$

By the boundedness of λ, Claim 1 is equivalent to the following claim.

Claim 2 $\liminf_{\tau\to\Lambda} diam_{\bar{g}\tau)}(M) > 0$.

Proof of Claim 2

Assume the contrary. Then we can find a sequence $\tau_k \to \Lambda, \tau_k > \Lambda_0$, such that $d_k \equiv diam_{\bar{g}(\tau_k)} \to 0$ and $d_k = \inf_{\Lambda_0 \le \tau \le \tau_k} diam_{\bar{g}(\tau)}(M)$. For each fixed k, we rescale $\bar{g}$ by the factor d_k^{-2} to obtain a solution $\bar{g}_k = d_k^{-2}\bar{g}$ of the volume normalized Ricci flow on $M \times [d_k^{-2}\Lambda_0, d_k^{-2}\Lambda]$ such that

$$\int_{d_k^{-2}\Lambda_0}^{d_k^{-2}\Lambda} \int_M |Rm|^{\frac{n+2}{2}} \le \delta_0 \tag{2.30}$$

holds for it. Because of (2.30) and the fact $diam_{\bar{g}_k}(M) \ge 1$ on $[d_k^{-2}\Lambda_0, d_k^{-2}\tau_k]$, we can apply the volume normalized part of Theorem A-1 to obtain a uniform estimate of $|Rm|$ for all $\bar{g}_k(d_k^{-2}\tau_k)$. Alternatively, since $diam_{\bar{g}_k}(M) \ge 1$ on $[d_k^{-2}\Lambda_0, d_k^{-2}\tau_k]$, the estimate (2.27) leads to the same uniform estimate. By the volume comparison, $\bar{g}_k(d_k^{-2}\tau_k)$ has uniformly bounded volume. But the volume of $\bar{g}_k(d_k^{-2}\tau_k)$ equals $d_k^{-\frac{n}{2}} vol_{g(0)}(M)$, which approaches ∞ as $k \to \infty$. This is a contradiction.

By Claim 1 and (2.26), we can apply Theorem A-1 to obtain a uniform estimate for $|Rm|$ over $[\frac{T_0+T}{2}, T)$ if T is finite, and over $[T_0+1, \infty)$ if $T = \infty$. Then the desired smooth convergences follow. ■

Proof of Theorem A-3

We choose $T_0 < T$ such that

$$\int_{T_0}^{T} \int_M |Rm|^{\frac{n+2}{2}} \le \delta_0. \tag{2.31}$$

Then we obtain uniform estimate of $|Rm|$ over the time interval $[\frac{T_0+T}{2}, T)$. The desired smooth convergences follow. The identification of the limit domain follows from an estimate of distance change based on the Ricci flow equation and the obtained curvature estimates. ■

3 Proofs of Theorems of Type B

Proof of Theorem B-1

The estimate (1.11) can be proved similarly to the corresponding part in [Proof of Theorem B, Ye3]. We handle the estimate (1.10) in more details. We focus on the Ricci flow as before. Assume that the estimate (1.10) fails to hold. Then we carry out the same construction as in the proof of Theorem A-1. We choose $\epsilon = \frac{1}{84}$ and $A = 10$. Then

$$d(x_0, \bar{x}, \bar{t}) < (2A+1)\epsilon < \frac{1}{4} \tag{3.1}$$

for g. By Claim 1 in the proof of Theorem A-1 we have

$$d(x_0, \bar{x}, t) \leq d(x_0, \bar{x}, \bar{t}) + 5Q^{-\frac{1}{2}} < \frac{1}{4} + 5\epsilon < \frac{1}{2} \tag{3.2}$$

for $t \in [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$. We also have $\frac{1}{3}Q^{-\frac{1}{2}} < \frac{1}{2}$. Consquently, (1.9) implies

$$\int_{\bar{t}-\frac{1}{2}\alpha_n Q^{-1}}^{\bar{t}} \frac{Q^{-\frac{n}{2}}}{vol_{g(t)}(B(\bar{x}, \frac{1}{3}Q^{-\frac{1}{2}}, t))} \int_{B(\bar{x}, \frac{1}{3}Q^{-\frac{1}{2}}, t)} |Rm|^{\frac{n+2}{2}} \leq 3^n \delta_0. \tag{3.3}$$

Hence we have for $\bar{g}$

$$\int_{-\frac{1}{2}\alpha_n}^{0} \frac{1}{vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{3}, t))} \int_{B(\bar{x}, \frac{1}{3}, t)} |Rm|^{\frac{n+2}{2}} \leq 3^n \delta_0. \tag{3.4}$$

As before, we also have for $\bar{g}$

$$|Rm|(\bar{x}, 0) = 1 \tag{3.5}$$

and

$$|Rm|(x, t) \leq 4 \tag{3.6}$$

whenever

$$-\frac{1}{2}\alpha_n \leq t \leq 0, d(\bar{x}, x, 0) \leq 1. \tag{3.7}$$

By [Theorem 4.1, An] and (3.6) we have

$$C_{S,2,\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t)) \leq \frac{C_5(n)}{vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{2}, t))^{\frac{1}{n}}} \tag{3.8}$$

for $t \in [-\frac{1}{2}\alpha_n, 0]$, with a positive constant $C_5(n)$ depending only on n. On the other hand, (3.6) implies

$$B(\bar{x}, \frac{1}{4}, t_1) \subset B(\bar{x}, \frac{1}{3}, t_2) \subset B(\bar{x}, \frac{1}{2}, t_3) \tag{3.9}$$

for all $t_1, t_2, t_3 \in [-\bar{\alpha}_n, 0]$, with a positive constant $\bar{\alpha}_n \leq \frac{1}{2}\alpha_n$ depending only on n. Consequently, we have

$$\int_{-\bar{\alpha}_n}^{0} \frac{1}{vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{3}, t))} \int_{B(\bar{x}, \frac{1}{4}, 0)} |Rm|^{\frac{n+2}{2}} \leq 3^n \delta_0 \tag{3.10}$$

and

$$C_{S,2,\bar{g}(t)}(B(\bar{x},\frac{1}{4},0)) \leq \frac{C_5(n)}{vol_{\bar{g}(t)}(B(\bar{x},\frac{1}{2},t)))^{\frac{1}{n}}} \tag{3.11}$$

for all $t \in [-\bar{\alpha}_n, 0]$. Moreover, (3.9) combined with (3.6) leads via the Ricci flow equation to

$$\min_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x},\frac{1}{2},t)) \geq e^{-4n(n-1)\bar{\alpha}_n} \max_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x},\frac{1}{3},t)) \tag{3.12}$$

for each $t \in [-\bar{\alpha}_n, 0]$. Now we apply [Theorem 2.1,Ye3] to deduce

$$\begin{aligned}
|Rm|(\bar{x},0) &\leq \frac{(1+\frac{2}{n})^{\frac{2\sigma_n}{n+2}} C_5(n)^{\frac{2n}{n+2}} C_6(n)}{\min\limits_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x},\frac{1}{2},t)))^{\frac{2}{n+2}}} \Big(\int_{-\bar{\alpha}_n}^0 \int\limits_{B(\bar{x},\frac{1}{4},0)} |Rm|^{\frac{n+2}{2}}\Big)^{\frac{2}{n+2}} \\
&\leq \frac{(1+\frac{2}{n})^{\frac{2\sigma_n}{n+2}} C_5(n)^{\frac{2n}{n+2}} C_6(n)}{\min\limits_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x},\frac{1}{2},t)))^{\frac{2}{n+2}}} \max_{-\bar{\alpha}_n \leq t \leq 0} vol_{\bar{g}(t)}(B(\bar{x},\frac{1}{3},t))^{\frac{2}{n+2}} \\
&\quad \cdot \Big(\int_{-\bar{\alpha}_n}^0 \frac{1}{vol_{\bar{g}(t)}(B(\bar{x},\frac{1}{3},t))} \int\limits_{B(\bar{x},\frac{1}{4},0)} |Rm|^{\frac{n+2}{2}}\Big)^{\frac{2}{n+2}} \\
&\leq 3^{\frac{2n}{n+2}}(1+\frac{2}{n})^{\frac{2\sigma_n}{n+2}} C_5(n)^{\frac{2n}{n+2}} C_6(n) \delta_0^{\frac{2}{n+2}} e^{\frac{2n(n-1)}{n+2}\bar{\alpha}_n},
\end{aligned} \tag{3.13}$$

with a suitable positive constant $C_6(n)$ depending only on n. Choosing

$$\delta_n = \frac{1}{3^n}(1+\frac{2}{n})^{-\sigma_n} C_5(n)^{-n} C_6(n)^{-\frac{n+2}{2}} e^{-n(n-1)\bar{\alpha}_n}$$

we then obtain $|Rm|(\bar{x},0) \leq \frac{1}{2}$, contradicting (3.5).

■

Proof of Theorem B-2

We handle the case of the Ricci flow. The other two cases are similar. We choose $T_0 < T$ such that

$$\int_{T_0}^T \sup_{x \in M, 0 < r \leq \frac{r_0}{2}} \frac{r^n}{vol_{g(t)}(B(x,r,t))} \int_{B(x,r,t)} |Rm|^{\frac{n+2}{2}}(\cdot,t) dvol_{g(t)} \leq \delta_0. \tag{3.14}$$

As in the proof of Theorem A-2, we rescale g to obtain a solution $\bar{g} = \bar{g}(\tau)$ of the volume normalized Ricci flow on $[0,\Lambda)$ in the case that M is closed. We have for $\bar{g}$

$$\int_{\Lambda_0}^{\Lambda} \sup_{x \in M, 0 < r \leq \frac{\bar{r}_0(t)}{2}} \frac{r^n}{vol_{\bar{g}(t)}(B(x,r,\tau))} \int_{B(x,r,\tau)} |Rm|^{\frac{n+2}{2}} \leq \delta_0, \tag{3.15}$$

where $\bar{r}_0(\tau) = \sqrt{\lambda(t(\tau))}r_0$.

By the proof of Claim 2 in the proof of Theorem A-2, we have

$$\lim_{\tau\to\Lambda} diam_{\bar{g}(\tau)}(M) > 0. \tag{3.16}$$

Claim $\lim_{t\to T} diam_{g(t)}(M) > 0$.

Proof of Claim

Assume the contrary, i.e.

$$\lim_{t\to T} diam_{g(t)}(M) = 0. \tag{3.17}$$

Then M is closed and we employ $\bar{g}$. Since $diam_{\bar{g}(\tau)(M)} = \sqrt{\lambda(t(\tau))}diam_{g(t(\tau))}(M)$, (3.17) and (3.16) imply that

$$\liminf_{\tau\to\Lambda} \bar{r}_0(\tau) > 0. \tag{3.18}$$

Case 1 $\Lambda = \infty$.

Then we have for $\bar{g}$

$$\lim_{\tau\to\infty}\int_\tau^\infty \sup_{x\in M, 0<r\leq\frac{\bar{r}_0(\tau)}{2}} \frac{r^n}{vol_{\bar{g}(\tau)}(B(x,r,\tau))}\int_{B(x,r,\tau)}|Rm|^{\frac{n+2}{2}} = 0. \tag{3.19}$$

On accout of (3.18), (3.16) and (3.19), we can apply (1.11) in Theorem B-1 to deduce that

$$\limsup_{\tau\to\infty}\sup_{x\in M}|Rm|(x,\tau) = 0. \tag{3.20}$$

It follows that $\bar{g}(\tau)$ subconverges smoothly to flat metrics g_∞ of volume $vol_{g(0)}(M)$ on M as $\tau \to \infty$. For each $\bar{t} > 0$ we rescale g by the factor $\lambda(\bar{t})$ to obtain $g_{\bar{t}}(t) = \lambda(\bar{t})g(\lambda(\bar{t})^{-1}t)$ on $[\lambda(\bar{t})\bar{t}, \lambda(\bar{t})T)$. Note that $g_{\bar{t}}$ is a solution of the Ricci flow with $g_{\bar{t}}(\lambda(\bar{t})\bar{t}) = \bar{g}(\tau(\bar{t}))$. As $\bar{t} \to T$, we have $\tau(\bar{t}) \to \infty$, and hence $g_{\bar{t}}(\lambda(\bar{t})\bar{t})$ subconverges smoothly to flat metrics of volume $vol_{g(0)}(M)$ on M. By Cheeger-Gromov compactness theorem and regularity of Einstein metrics, the space of flat metrics on M of volume $vol_{g(0)}(M)$ modulo the diffeomorphism group of M is compact with respect to the smooth topology. Hence we can apply the stability theorem in [GIK] to conclude that $g_{\bar{t}}(t)$ extends to a smooth solution of the Ricci flow for all time $t \in [\lambda(\bar{t})\bar{t}, \infty)$ and converges smoothly to a flat metric as $t \to \infty$, provided that $\bar{t}$ is close enough to T. But this contradicts (3.17).

Case 2 $\Lambda < \infty$.

By (3.15), (3.16) and (3.18), we can apply Theorem B-1 to obtain a uniform estimate of $|Rm|$ for $\bar{g}$ over $[0, \lambda)$. In particular, the scalar curvature is unformly bounded. It follows that the scaling factor λ satisfies $a < \lambda < b$ for two positive constants a and b. But this leads to a contradiction on account of (3.17) and (3.16). Hence the claim is proved.

Given the claim and (3.14), it is clear that we can apply Theorem B-1 to obtain a uniform upper bound for $|Rm|$ over $[0, T)$. The desired smooth convergence follows. ■

Proof of Theorem B-3

This is similar to the proof of Theorem A-3. ■

4 Proofs of Theorems of Type C

Proof of Theorem C-1

We establish the condition (1.9). Then the theorem follows from Theorem B-1. By rescaling we can assume $r_0 = 1$. Then (1.14) becomes

$$Ric \geq -(n-1)g. \tag{4.1}$$

By (1.15), we have now

$$\int_0^T \frac{1}{vol_{g(t)}(B(x_0, 1, t))} \int_{B(x_0,1,t)} |Rm|^{\frac{n+2}{2}} dvol_{g(t)} \leq \delta_0. \tag{4.2}$$

By Bishop-Gromov relative volume comparison, we have

$$vol_{g(t)}(B(x, R, t)) \leq \frac{v_{-1}(R)}{v_{-1}(r)} vol_{g(t)}(B(x, r, t)) \leq C(n) \frac{vol_{g(t)}(B(x, r, t))}{r^n}, \tag{4.3}$$

with a positive constant $C(n)$ depending only on n, provided that $t \in [0, T], d(x_0, x, t) < 1$, and $0 < r < R \leq 1 - d(x_0, x, t)$. Here $v_{-1}(r)$ denotes the volume of a geodesic ball of radius r in $\mathbf{H}^n$, the n-dimensional hyperbolic space (of sectional curvature -1). If $t \in [0, T], d(x_0, x, t) \leq \frac{1}{4}$, we then have $B(x_0, \frac{1}{4}, t) \subset B(x, \frac{1}{2}, t) \subset B(x_0, 1, t)$. Consequently,

$$\begin{aligned} vol_{g(t)}(B(x, r, t)) &\geq C(n)^{-1} r^n vol_{g(t)}(B(x, \frac{1}{2}, t)) \geq C(n)^{-1} r^n vol_{g(t)}(B(x_0, \frac{1}{4}, t)) \\ &\geq 4^{-n} C(n)^{-2} r^n vol_{g(t)}(B(x_0, 1, t)) \end{aligned} \tag{4.4}$$

for $0 < r \leq \frac{1}{2}$. Hence we infer

$$\frac{r^n}{vol_{g(t)}(B(x,r,t))}\int_{B(x,r,t)}|Rm|^{\frac{n}{2}}dvol_{g(t)} \leq \\ \frac{4^nC(n)^2}{vol_{g(t)}(B(x_0,1,t))}\int_{B(x_0,1,t)}|Rm|^{\frac{n}{2}}dvol_{g(t)}, \tag{4.5}$$

whenever $0 \leq t < T, 0 < r \leq \frac{1}{2}$, and $d(x_0,x,t) \leq \frac{1}{4}$. It follows that

$$\int_0^T \frac{r^n}{vol_{g(t)}(B(x,r,t))^*}\int_{B^*(x,r,t)}|Rm|^{\frac{n}{2}}dvol_{g(t)} \leq \\ 4^nC(n)^2\int_0^T \frac{1}{vol_{g(t)}(B(x_0,1,t))}\int_{B(x_0,1,t)}|Rm|^{\frac{n}{2}}dvol_{g(t)} \leq 4^nC(n)^2\delta_0 \tag{4.6}$$

for all $x \in M$. Choosing δ_0 to be the δ_0 in Theorem B-1 multiplied by $4^{-n}C(n)^{-2}$ and replacing r_0 by $\frac{r_0}{2}$ we then have all the conditions of Theorem B-1. The desired estimate follows. By the proof of Theorem B-1, (1.10) actually holds with $\epsilon_0 = \frac{1}{84}$. Hence we obtain $\epsilon_0 = \frac{1}{168}$ now.

■

Proofs of Theorem C-2 and Theorem C-3

These are similar to proofs of Theorem B-2 and Theorem B-3.

■

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