Midterm Exam
Math 122B
Summer 2011
Prof. Rick Ye

Your Name:
Your Perm Number:
Your Signature:

1.
2.
3.
4.
Total Score:

1. **Be sure to write down all the steps.**

2. **Write cleanly and clearly!**

1. (33 points) Find the Maclaurin series expansion of the function

$$f(z) = \frac{2z+1}{z^3+8}.$$

Determine the radius of convergence of the series.

Solution By the convergence of the geometric series we have for $|z|^3 < 8$, i. e. $|z| < 2$

$$\begin{aligned}
\frac{1}{z^3+8} &= \frac{1}{8} \cdot \frac{1}{1+\frac{z^3}{8}} = \\
&= \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z^3}{8} \right)^n \\
&= \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \frac{z^{3n}}{2^{3n}}. \qquad (0.1)
\end{aligned}$$

It follows that

$$\begin{aligned}
f(z) &= (2z+1) \cdot \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \frac{z^{3n}}{2^{3n}} \\
&= \frac{1}{8} \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{3n+1}}{2^{3n-1}} + \sum_{n=0}^{\infty} (-1)^n \frac{z^{3n}}{2^{3n}} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{3n+2}} (z^{3n+1} + \frac{1}{2} z^{3n}). \qquad (0.2)
\end{aligned}$$

By the uniqueness of Taylor series, this is the Maclaurin series of f and converges for $|z| < 2$. On the other hand, $f(z)$ is not analytic at roots of $z^3 + 8 = 0$ which have modulus 2. By the Taylor series theorem we conclude that the convergence radius of the Maclaurin series cannot be larger than 2. Hence its convergence radius is 2. One can also determine the convergence radius directly by the ratio test or the root test. ∎

2. (33 points) Find the Taylor series of the function $f(z) = z \log z$ around $z_0 = -1$, where we use the following branch of $\log z$:

$$\log z = \ln r + i\theta$$

for $z = re^{i\theta}$ with $0 < \theta < 2\pi$. In what region does the Taylor series converge to $f(z)$? Why?

Solution There holds $-1 = e^{i\pi}$. Hence

$$f(-1) = (-1) \log(-1) = -i\pi. \qquad (0.3)$$

On the other hand, there hold

$$f'(z) = \log z + 1, f''(z) = z^{-1}, f'''(z) = -z^{-2} \tag{0.4}$$

and in general

$$f^{(k+2)}(z) = (-1)^{k+2}k!z^{-(k+1)} \tag{0.5}$$

for $k = 0, 1, 2, ...$ We infer

$$f'(-1) = 1 + i\pi \tag{0.6}$$

and

$$f^{(k+2)}(-1) = (-1)^{(k+2)-(k+1)}k! = -k!. \tag{0.7}$$

The Taylor series of f around -1 is then given by

$$\begin{aligned} & -i\pi + (1+i\pi)(z+1) - \sum_{n=0}^{\infty} \frac{n!}{(n+2)!}(z+1)^{n+2} \\ = \quad & -i\pi + (1+i\pi)(z+1) - \sum_{n=0}^{\infty} \frac{(z+1)^{n+2}}{(n+1)(n+2)}. \end{aligned} \tag{0.8}$$

By Taylor series theorem, it converges to f in the largest disc of center 0 in which f is analytic. (It diverges on the outside of this disc.) Since f is analytic precisely in the complement of its branch cut $\{x+yi : x \geq 0, y = 0\}$, we conclude that the Taylor series of f converges to it in the disc $\{|z+1| < 1\}$. ■

3. (34 points) 1) Determine the annuli around the origin in which the function $f(z) = \frac{z+1}{(z-1)(z+5)}$ is analytic. Note: Annuli include punctured discs and annuli with infinite outer radius.
2) Find the Laurent series of f in each of the annuli.
(Note: you need to find three annuli and then three Laurent series.)

Solution 1) f has two singularities: $z_1 = 1$ and $z_2 = -5$. Hence the desired annuli are the following ones: $\{0 < |z| < 1\}, \{1 < |z| < 5\}$ and $\{|z| > 5\}$.
2) There holds

$$\frac{A}{z-1} + \frac{B}{z+5} = \frac{(A+B)z + 5A - B}{(z-1)(z+5)}. \tag{0.9}$$

Setting $A + B = 1, 5A - B = 1$ we deduce $A = 1/3, B = 2/3$. Hence

$$\frac{z+1}{(z-1)(z+5)} = \frac{1/3}{z-1} + \frac{2/3}{z+5}. \tag{0.10}$$

Now we determine the Laurent series in each of the three annuli.
a) In $\{0 < |z| < 1\}$ there holds

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0} z^n, \; \frac{1}{z+5} = \frac{1}{5} \cdot \frac{1}{1+z/5} = \frac{1}{5}\sum_{n=0}(-1)^n \frac{z^n}{5^n}. \tag{0.11}$$

It follows that

$$\begin{aligned} f(z) &= -\frac{1}{3}\sum_{n=0} z^n + \frac{2}{15}\sum_{n=0}(-1)^n \frac{z^n}{5^n} \\ &= \sum_{n=0} \frac{1}{3}(\frac{2(-1)^n}{5^{n+1}} - 1) z^n. \end{aligned} \tag{0.12}$$

b) In $\{1 < |z| < 5\}$ there holds

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-1/z} = \frac{1}{z} \cdot \sum_{n=0} \frac{1}{z^n} = \sum_{n=0} \frac{1}{z^{n+1}}. \tag{0.13}$$

On the other hand, the above expansion for $1/(z+5)$ continues to hold. Hence we deduce

$$f(z) = \frac{1}{3}\sum_{n=1} \frac{1}{z^n} + \frac{2}{15}\sum_{n=0}(-1)^n \frac{z^n}{5^n}. \tag{0.14}$$

c) In $\{|z| > 5\}$ the expansion for $1/(z-1)$ obtained in b) continues to hold. On the other hand, we have

$$\frac{1}{z+5} = \frac{1}{z} \cdot \frac{1}{1+z/5} = \frac{1}{z} \cdot \sum_{n=0}(-1)^n \frac{z^n}{5^n} = \frac{1}{z} + \sum_{n=0}(-1)^n \frac{z^n}{5^{n+1}}. \tag{0.15}$$

It follows that

$$\begin{aligned} f(z) &= \frac{1}{3}\sum_{n=1} \frac{1}{z^n} + \frac{2}{3} \cdot \frac{1}{z} + \frac{2}{3}\sum_{n=0}(-1)^n \frac{z^n}{5^{n+1}} \\ &= \frac{1}{z} + \frac{1}{3}\sum_{n=2} \frac{1}{z^n} + \frac{2}{3}\sum_{n=0}(-1)^n \frac{z^n}{5^{n+1}}. \end{aligned} \tag{0.16}$$