Final Exam
Math 240 B
Winter 2011
Prof. Ye

Your name
Your perm. number

Scores
1.
2.
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Total:

Each problem is worth 16 points. In particular, 2 extra credit points are included.

Please email your final and homework to Prof. Ye:
yer@math.ucsb.edu
You can type up or scan your documents. The final is due on Saturday, March 19, 2010. Please keep your originals.

1. Let M be a smooth manifold and $\alpha \in \Omega^m(M)$, i. e. α is a smooth form of degree m on M.
1) Assume $m = 1$. Prove the following formula for the exterior derivative

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) \tag{0.1}$$

for smooth vector fields X and Y.
2) Prove the following general formula for the exterior derivative

$$\begin{aligned} d\alpha(X_1, ..., X_{m+1}) &= \sum_{i=1}^{m+1}(-1)^{i-1}X_i\alpha(X_1, ..., \hat{X}_i, ..., X_{m+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{m+1}) \end{aligned}$$

for arbitrary smooth vector fields X_i, where the hat means that the term is omitted.

Note: In John Lee's book, a proof of this formula is given. You are required to find a proof without reading Lee's proof, until you have become completely frustrated (if that happens at all).

2. Consider a smooth manifold M of dimension n. Let $\Omega^*(M) = \oplus_{0\le k\le n}\Omega^k(M)$ denote the algebra of smooth differential forms on M. Let $\Omega_c^*(M) = \oplus_{0\le k\le n}\Omega_c^k(M)$ denote the subalgebra of $\Omega(M)$ consisting of smooth differential forms on M with compact support.
1) Show that the De Rham cohomology $H^*_{dR,c}(M)$ with compact suppot can be defined analogously to $H^*_{dR}(M)$.
2) Show tha the inclusion map of $\Omega_c^*(M)$ into $\Omega^*(M)$ induces a natural homomorphism $F : H^*_{dR,c}(M) \to H^*_{dR}$.
3) Show that F is not injective for $M = \mathbf{R}$ (the real line).
4) Assume that M is noncompact. Show that the restriction of F to $H^k_{dR,c}(M)$ is not injective for each $0 < k \le n$. Hint: Use Stokes Theorem.
5) Assume that M is connected and noncompact. Show that $H^0_{dR,c}(M) = \{0\}$.

3. 1) Show that the 1-dimensional De Rham cohomology group $H^1_{dR}(S^1 \times S^1)$ is isomorphic to $\mathbf{R}^2$.
2) Let $n \ge 2$ be general. Show that the 1-dimensional De Rham cohomology group $H^1_{dR}(T^n)$ of the n-dimensional torus $T^n = S^1 \times ... \times S^1$ (n factors) is isomorphic to $\mathbf{R}^n$
Hint: Extend the proof for the case $n = 1$ presented in the lectures.

4. Let ∇ be a connection on the tangent bundle of a smooth manifold M. Recall that it extends to all tensor fields, i. e. we can take covariant derivatives $\nabla_v\sigma$ for smooth tensor fields σ. Let $\gamma : [a, b] \to M$ be a smooth curve, and $\sigma = \sigma(t)$ a smooth tensor

field of some type (r,s) along γ, i. e. $\sigma(t) \in T^r_s(T_{\gamma(t)}M)$ for each t.
1) We define the covariant derivative $\nabla_{\frac{d}{dt}}\sigma$ as follows. Let $[a',b']$ be a subinterval of $[a,b]$ such that $\gamma([a',b'])$ is contained in a coordinate chart (U,ϕ). Let $\sigma^{i_1...i_r}_{j_1...j_s}$ be basis tensor fields in the chart, i. e.

$$\sigma^{i_1...i_r}_{j_1...j_s} = dx^{i_1} \otimes ... \otimes dx^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \otimes ... \otimes \frac{\partial}{\partial x^{j_s}}. \tag{0.2}$$

We have

$$\sigma(t) = \sum a^{j_1...j_s}_{i_1...i_r}(t)\sigma^{i_1...i_r}_{j_1...j_s}|_{\gamma(t)} \tag{0.3}$$

for some smooth functions $a^{j_1...j_s}_{i_1...i_r}(t)$. Then we set for $a' \le t \le b'$

$$\nabla_{\frac{d}{dt}} = \sum(\frac{d}{dt}a^{j_1...j_s}_{i_1...i_r}(t))\sigma^{i_1...i_r}_{j_1...j_s}|_{\gamma(t)} + \sum a^{j_1...j_s}_{i_1...i_r}(t)\nabla_{\gamma'(t)}\sigma^{i_1...i_r}_{j_1...j_s}. \tag{0.4}$$

Show that this definition is independent of the choice of the chart.
2) Let $\sigma_0 \in T^r_s(T_{\gamma(a)}M)$ for given r and s. Show that there is a *unique* parallel smooth tensor field σ of type (r,s) along γ, such that $\sigma(0) = \sigma_0$. Here "parallel" means $\nabla_{\frac{d}{dt}} \equiv 0$. ($\sigma(t)$ is called the parallel transport of σ_0 along the curve γ.)

Hint: First handle the case of type $(0,1)$, i. e. the case of vector fields. This special case is worth more than half of the credit.

5. Let (M,g,∇) be a Riemannian manifold together with the Levi-Civita connection of g.
1) Let $\gamma : [a,b] \to M$ be a smooth curve and $\{v_1,...,v_n\}$ an orthonormal basis of the tangent space at $\gamma(a)$. Let $e_i(t)$ be the parallel transport of v_i along γ. Show that $\{e_i(t)\}$ is orthonormal for each t.
2) Assume in addition that M is oriented and $(v_1,...,v_n)$ is a positive basis. Show that $(e_1(t),...,e_n(t))$ is a positive basis for each t.

6. Let (M,g,∇) be an oriented Riemannian manifold together with the Levi-Civita connection of g.
1) Show that g is parallel, i. e. $\nabla_v g = 0$ at each point $p \in M$ and for each tangent vector $v \in T_pM$.
2) Show that the volume form $dvol$ is parallel, i. e. $\nabla_v dvol = 0$ at each point $p \in M$ and for each tangent vector $v \in T_pM$.

7. Assume the following result: $H^n_{dR}(M)$ is isomorphic to $\mathbf{R}$ for a connect, closed and orientable smooth manifold M of dimension n. Prove that $H^n_{dR}(M) = \{0\}$ for a connected, closed and non-orientable manifold M of dimension n. Hint: utilize the oriented double cover of M.