Midterm Exam
Math 117
Summer 2014
Prof. R. Ye

Your Name:
Your Signature:
Your Perm Number:

Scores:
1.
2.
3.
4.
Total: (out of 100)

Please present detailed steps of your solutions.

Write letters in larger size, and press harder when writing with pencils!

1. (25 points) Prove that

$$\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

for all $n \in \mathbb{N}$.

Solution Let $A(n)$ denote the left hand side of the above equation, and let $B(n)$ denote its right hand side. For $n = 1$ we have

$$A(n) = A(1) = \frac{1}{(1+1)!} = \frac{1}{2} \tag{0.1}$$

and

$$B(n) = B(1) = 1 - \frac{1}{(1+1)!} = 1 - \frac{1}{2} = \frac{1}{2}. \tag{0.2}$$

Hence there holds $A(1) = B(1)$.

Now we assume $A(n) = B(n)$ for a natural number n. There holds

$$\begin{aligned} A(n+1) &= A(n) + \frac{n+1}{(n+2)!} = B(n) + \frac{n+1}{(n+2)!} \\ &= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \\ &= 1 - \frac{(n+2)-(n+1)}{(n+2)!} = 1 - \frac{1}{(n+2)!} \\ &= B(n+1). \end{aligned} \tag{0.3}$$

By mathematical induction we conclude that $A(n) = B(n)$ for all natural number n. ■

4. (25 points) Assume $x \geq 0$ and $x < y$ for all $y > 0$. Prove that $x = 0$.

Solution We argue by contradiction. Assume that $x > 0$. Then $\frac{1}{2}x > 0$.
(Note that $1 > 0$, see Problem 2 below. If $\frac{1}{2} < 0$, then $1 = \frac{1}{2} + \frac{1}{2} < \frac{1}{2}$ by O3 and A4. This contradicts the fact that $1 > 0$, see Problem 2 below. Hence $\frac{1}{2} > 0$. By O4 and Theorem 11.1(b) we then deduce $\frac{1}{2}x > 0$.)
But $x - \frac{1}{2}x = (1 - \frac{1}{2})x = \frac{1}{2}x > 0$. It follows that $x > \frac{1}{2}x$. Setting $y = \frac{1}{2}x$ we get a contradiction to the assumption that $x < y$ for all $y > 0$. Since $x \geq 0$, we infer that $x = 0$. ■

2. Part 1. (25 points) Prove the following
(1) $1 > 0$.
(2) If $x < 0$, then $1/x < 0$.

Solution
(1) Assume $1 < 0$. Then $1 + (-1) < 0 + (-1)$, and hence $0 < -1$. Based on O4, we multiply $1 < 0$ with -1 to deduce $(-1)1 < (-1)0$, and hence $-1 < 0$. (Note that $(-1)0 = 0$ by Theorem 11.1(b). By M4, there holds $(-1)1 = -1$.) This is a controdiction. Since $1 \neq 0$ (by M4), we infer that $-1 < 0$. ■

(2) Assume that $1/x > 0$. Based on O4, we multiply $x < 0$ by $1/x$ to deduce $1/x \cdot x < 1/x \cdot 0$, and hence $1 < 0$ (by Theorem 11.1(b)), contradicting (1). We conclude that $1/x < 0$. ■

2. Part 2. (25 points) Prove the following
(3) If $x < y$ and $z < 0$, then $zx > zy$.
(4) If $x < y < 0$, then $1/x > 1/y$.

Solution
(3) Assume $x < y$ and $z < 0$. Since $z < 0$, there holds $(-z) + z < -z$ (by O3 and A4), and hence $0 < -z$ (by A5). By O4 we then deduce $x(-z) < y(-z)$. Since $-w = (-1)w$ for all $w \in \mathbf{R}$ (by Theorem 11.1(c)), we infer by M3 and M2 that $-xz < -yz$. Hence $yz + xz - xz < yz + xz - yz$ by O3. It follows that $yz < xz$. ■

(4) Assume $x < y < 0$. By 2(2) we then have $1/x > 0$ and $1/y > 0$, and hence $1/x \cdot 1/y > 0$ (by O4 and Theorem 11.1(b)). By O4 we then deduce

$$\frac{1}{x} \cdot \frac{1}{y} \cdot x < \frac{1}{x} \cdot \frac{1}{y} \cdot y. \tag{0.4}$$

(M3 is employed here.) It follows that $1/y < 1/x$, i.e. $1/x > 1/y$. (We employ here M2.) ■