3

The Real Numbers

In this chapter we present in some detail many of the important properties of the set $\mathbb{R}$ of real numbers. The real numbers form the background for virtually all the analysis we do in this text. Our approach will be axiomatic, but not constructive. That is, we shall not construct the real numbers from some simpler (?) set such as the natural numbers. Instead, we shall assume the existence of $\mathbb{R}$ and postulate the properties that characterize it.[†]

We begin in Section 10 by looking at the natural numbers and mathematical induction. In Section 11 we consider the field and order axioms that begin to characterize $\mathbb{R}$. The completeness axiom in Section 12 is the final axiom and deserves special attention because of its central role in the rest of analysis. In Sections 13 and 14 we develop some of the topological properties of the reals that will be useful in describing the behavior of sequences and functions. In Section 15 (an optional section) we look at these properties in the more general context of a metric space.

Section 10 NATURAL NUMBERS AND INDUCTION

In Section 5 we agreed to let $\mathbb{N}$ denote the set of positive integers, also called the natural numbers. Thus

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

[†] For a constructive approach to developing the reals from the rationals, the rationals from the natural numbers, and the natural numbers from basic set theory, see Henkin and others (1962), Stewart and Tall (1977), or Hamilton (1982). See also Exercises 10.21, 11.12, and 11.13.

It is possible to develop all the properties (and even the existence) of the natural numbers in a rigorous way from set theory and a few additional axioms. But since our discussion of set theory was not entirely rigorous, there is little to be gained by going through the steps of that development here. Rather, we shall assume that the reader is familiar with the usual arithmetic operations of addition and multiplication and with the notion of what it means for one natural number to be less than another.

There is one additional property of $\mathbb{N}$ that we shall assume as an axiom. (That is, we accept it as true without proof.) It expresses in a precise way the intuitive idea that each nonempty subset of $\mathbb{N}$ must have a least element.

10.1 AXIOM (Well-Ordering Property of $\mathbb{N}$) If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \leq k$ for all $k \in S$.

One important tool to be used when proving theorems about the natural numbers is the principle of mathematical induction. It enables us to conclude that a given statement about natural numbers is true for all the natural numbers without having to verify it for each number one at a time (which would be an impossible task!).

10.2 THEOREM (Principle of Mathematical Induction) Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$, provided that

(a) $P(1)$ is true, and
(b) for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Proof: The strategy of our argument will be a proof by contradiction using tautology (g) in Example 3.12. That is, we suppose that (a) and (b) hold but that $P(n)$ is false for some $n \in \mathbb{N}$. Let

$$S = \{n \in \mathbb{N} : P(n) \text{ is false}\}.$$

Then S is not empty and the well-ordering property guarantees the existence of an element $m \in S$ that is a least element of S. Since $P(1)$ is true by hypothesis (a), $1 \notin S$, so that $m > 1$. It follows that $m - 1$ is also a natural number, and since m is the least element in S, we must have $m - 1 \notin S$.

But since $m - 1 \notin S$, it must be that $P(m-1)$ is true. We now apply hypothesis (b) with $k = m - 1$ to conclude that $P(k+1) = P(m)$ is true. This implies that $m \notin S$, which contradicts our original choice of m. ♦

It is customary to refer to the verification of part (a) of Theorem 10.2 as the **basis for induction** and part (b) as the **induction step**. The assumption that $P(k)$ is true in verifying part (b) is known as the **induction hypothesis**. It is essential that both parts be verified to have a valid proof using mathematical induction. In practice, it is usually the induction step that is the more difficult part.

10.3 EXAMPLE Prove that $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ for every natural number n.

Proof: Let $P(n)$ be the statement

$$1+2+3+\cdots+n = \tfrac{1}{2}n(n+1).$$

Then $P(1)$ asserts that $1 = \frac{1}{2}(1)(1+1)$, $P(2)$ asserts that $1+2 = \frac{1}{2}(2)(2+1)$, and so on. In particular, we see that $P(1)$ is true, and this establishes the basis for induction.

To verify the induction step, we suppose that $P(k)$ is true, where $k \in \mathbb{N}$. That is, we assume

$$1+2+3+\cdots+k = \tfrac{1}{2}k(k+1).$$

Since we wish to conclude that $P(k+1)$ is true, we add $k+1$ to both sides to obtain

$$\begin{aligned}1+2+3+\cdots+k+(k+1) &= \tfrac{1}{2}k(k+1)+(k+1)\\ &= \tfrac{1}{2}[k(k+1)+2(k+1)]\\ &= \tfrac{1}{2}(k+1)(k+2)\\ &= \tfrac{1}{2}(k+1)[(k+1)+1].\end{aligned}$$

Thus $P(k+1)$ is true whenever $P(k)$ is true, and by the principle of mathematical induction, we conclude that $P(n)$ is true for all n. ♦

Since the format of a proof using mathematical induction always consists of the same two steps (establishing the basis for induction and verifying the induction step), it is common practice to reduce some of the formalism by omitting explicit reference to the statement $P(n)$. It is also acceptable to omit identifying the steps by name, but we must be certain that they are both actually there.

10.4 EXAMPLE Prove by induction that $7^n - 4^n$ is a multiple of 3, for all $n \in \mathbb{N}$.

Proof: Clearly, this is true when $n = 1$, since $7^1 - 4^1 = 3$. Now let $k \in \mathbb{N}$ and suppose that $7^k - 4^k$ is a multiple of 3. That is, $7^k - 4^k = 3m$ for some

$m \in \mathbb{N}$. It follows that[†]

$$\begin{aligned} 7^{k+1} - 4^{k+1} &= 7^{k+1} - 7\cdot 4^k + 7\cdot 4^k - 4\cdot 4^k \\ &= 7(7^k - 4^k) + 3\cdot 4^k \\ &= 7(3m) + 3\cdot 4^k \\ &= 3(7m + 4^k). \end{aligned}$$

Since m and k are natural numbers, so is $7m + 4^k$. Thus $7^{k+1} - 4^{k+1}$ is also a multiple of 3, and by induction we conclude that $7^n - 4^n$ is a multiple of 3 for all $n \in \mathbb{N}$. ♦

10.5 PRACTICE Observe that

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2. \end{aligned}$$

Figure out a general formula and prove your answer using mathematical induction.

There is a generalization of the principle of mathematical induction that enables us to conclude that a given statement is true for all natural numbers sufficiently large. More precisely, we have the following:

10.6 THEOREM Let $m \in \mathbb{N}$ and let $P(n)$ be a statement that is either true or false for each $n \geq m$. Then $P(n)$ is true for all $n \geq m$, provided that

(a) $P(m)$ is true, and
(b) for each $k \geq m$, if $P(k)$ is true, then $P(k+1)$ is true.

[†] In the proof we have added and subtracted the term $7\cdot 4^k$. Where did it come from? We want somehow to use the induction hypothesis that $7^k - 4^k = 3m$, so we break 7^{k+1} apart into $7\cdot 7^k$. We would like to have $7^k - 4^k$ as a factor instead of just 7^k, but to do this we must subtract (and add) the term $7\cdot 4^k$.

Alternatively, we could have broken 4^{k+1} apart into $4\cdot 4^k$. This time to replace 4^k by $7^k - 4^k$, we must add (and subtract) $4\cdot 7^k$:

$$\begin{aligned} 7^{k+1} - 4^{k+1} &= 7^{k+1} - 4\cdot 7^k + 4(7^k - 4^k) \\ &= 7^k(7-4) + 4(7^k - 4^k) \\ &= 7^k\cdot 3 + 4(3m) = 3(7^k + 4m). \end{aligned}$$

Once again we see that $7^{k+1} - 4^{k+1}$ is a multiple of 3.

Proof: The proof will use the original principle of induction (Theorem 10.2). For each $r \in \mathbb{N}$, let $Q(r)$ be the statement "$P(r+m-1)$ is true." Then from (a) we know that $Q(1)$ holds. Now let $j \in \mathbb{N}$ and suppose that $Q(j)$ holds. That is, $P(j+m-1)$ is true. Since $j \in \mathbb{N}$,

$$j+m-1 = m+(j-1) \geq m,$$

so by (b), $P(j+m)$ must be true. Thus $Q(j+1)$ holds and the induction step is verified. We conclude that $Q(r)$ holds for all $r \in \mathbb{N}$.

Now if $n \geq m$, let $r = n - m + 1$, so that $r \in \mathbb{N}$. Since $Q(r)$ holds, $P(r+m-1)$ is true. But $P(r+m-1)$ is the same as $P(n)$, so $P(n)$ is true for all $n \geq m$. ♦

Review of Key Terms in Section 10

Well-ordering property of $\mathbb{N}$	Induction step
Principle of mathematical induction	Induction hypothesis
Basis for induction	

ANSWERS TO PRACTICE PROBLEMS

10.5 The general formula is

$$1+3+5+\cdots+(2n-1) = n^2,$$

and we have already seen that this is true for $n = 1$. For the induction step, suppose that $1+3+5+\cdots+(2k-1) = k^2$. Then

$$\begin{aligned} 1+3+5+\cdots+(2k-1)+(2k+1) &= k^2+(2k+1) \\ &= (k+1)^2. \end{aligned}$$

Since this is the formula for $n = k+1$, we conclude by induction that the formula holds for all $n \in \mathbb{N}$. ♦

EXERCISES

*Exercises marked with * are used in later sections and exercises marked with ☆ have hints or solutions in the back of the book.*

10.1 Mark each statement True or False. Justify each answer.

(a) If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \geq k$ for all $k \in S$.

(b) The principle of mathematical induction enables us to prove that a statement is true for all natural numbers without directly verifying it for each number.

10.2 Mark each statement True or False. Justify each answer.

(a) A proof using mathematical induction consists of two parts: establishing the basis for induction and verifying the induction hypothesis.

(b) Suppose m is a natural number greater than 1. To prove $P(k)$ is true for all $k \geq m$, we must first show that $P(k)$ is false for all k such that $1 \leq k < m$.

***10.3** Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

***10.4** Prove that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

10.5 Prove that $1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2$ for all $n \in \mathbb{N}$.☆

***10.6** Prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \text{ for all } n \in \mathbb{N}.$$

***10.7** Prove that $1 + r + r^2 + \cdots + r^n = (1 - r^{n+1})/(1 - r)$ for all $n \in \mathbb{N}$, when $r \neq 1$. ☆

***10.8** Prove that

$$\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots + \frac{1}{4n^2 - 1} = \frac{n}{2n+1}, \text{ for all } n \in \mathbb{N}.$$

10.9 Prove that $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$, for all $n \in \mathbb{N}$.

10.10 Prove that $1(1!) + 2(2!) + \cdots + n(n!) = (n + 1)! - 1$, for all $n \in \mathbb{N}$.

10.11 Prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}, \text{ for all } n \in \mathbb{N}.$$

10.12 Prove that $1 + 2\cdot 2 + 3\cdot 2^2 + \cdots + n2^{n-1} = (n-1)2^n + 1$, for all $n \in \mathbb{N}$.

10.13 Prove that $5^{2n} - 1$ is a multiple of 8 for all $n \in \mathbb{N}$.☆

10.14 Prove that $9^n - 4^n$ is a multiple of 5 for all $n \in \mathbb{N}$.

10.15 Indicate what is wrong with each of the following induction "proofs."

(a) **Theorem:** For each $n \in \mathbb{N}$, let $P(n)$ be the statement "Any collection of n marbles consists of marbles of the same color." Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Clearly, $P(1)$ is a true statement. Now suppose that $P(k)$ is a true statement for some $k \in \mathbb{N}$. Let S be a collection of $k+1$ marbles. If one marble, call it x, is removed, then the induction hypothesis applied to the remaining k marbles implies that these k marbles all have the same color. Call this color C. Now if x is returned to the set S and a different marble is removed, then again the remaining k marbles must all be of the same color C. But one of these marbles is x, so in fact all $k+1$ marbles have the same color C. Thus $P(k+1)$ is true, and by induction we conclude that $P(n)$ is true for all $n \in \mathbb{N}$. ♦

(b) **Theorem:** For each $n \in \mathbb{N}$, let $P(n)$ be the statement "$n^2 + 7n + 3$ is an even integer." Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: Suppose that $P(k)$ is true for some $k \in \mathbb{N}$. That is, $k^2 + 7k + 3$ is an even integer. But then

$$\begin{aligned}(k+1)^2 + 7(k+1) + 3 &= (k^2 + 2k + 1) + 7k + 7 + 3 \\ &= (k^2 + 7k + 3) + 2(k + 4),\end{aligned}$$

and this number is even, since it is the sum of two even numbers. Thus $P(k + 1)$ is true. We conclude by induction that $P(n)$ is true for all $n \in \mathbb{N}$. ♦

10.16 Prove that $2+5+8+\cdots+(3n-1) = \frac{1}{2}n(3n+1)$ for all $n \in \mathbb{N}$.

10.17 Conjecture a formula for the sum $5 + 9 + 13 + \cdots + (4n + 1)$, and prove your conjecture using mathematical induction. ☆

10.18 Prove that

$$(2)(6)(10)(14)\cdots(4n-2) = \frac{(2n)!}{n!}, \text{ for all } n \in \mathbb{N}.$$

10.19 Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$, for all $n \in \mathbb{N}$ with $n \geq 2$.

10.20 Prove that $(\cos x + i \sin x)^n = \cos(nx) + i\sin(nx)$, for all $n \in \mathbb{N}$, where $i = \sqrt{-1}$. You may use the identities $\cos(a + b) = \cos a \cos b - \sin a \sin b$ and $\sin(a + b) = \sin a \cos b + \cos a \sin b$.

10.21 Indicate for which natural numbers n the given inequality is true. Prove your answers by induction.

(a) $n^2 \leq n!$ ☆
(b) $n^2 \leq 2^n$
(c) $2^n \leq n!$

***10.22** Use induction to prove Bernoulli's inequality: If $1 + x > 0$, then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

10.23 Prove that for all integers $x \geq 8$, x can be written in the form $3m + 5n$, where m and n are non-negative integers. ☆

10.24 Consider the statement: "For all integers $x \geq k$, x can be written in the form $5m + 7n$, where m and n are non-negative integers."

(a) Find the smallest value of k that makes the statement true.
(b) Prove the statement is true with k as in part (*a*).

***10.25** Prove the principle of strong induction: Let $P(n)$ be a statement that is either true or false for each $n \in \mathbb{N}$. Then $P(n)$ is true for all $n \in \mathbb{N}$ provided that

(a) $P(1)$ is true, and
(b) for each $k \in \mathbb{N}$, if $P(j)$ is true for all integers j such that $1 \leq j \leq k$, then $P(k+1)$ is true. ☆

10.26 Prove that for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $n \leq k^2 \leq 2n$.

10.27 In the song "The Twelve Days of Christmas," gifts are sent on successive days according to the following pattern:

First day: A partridge in a pear tree.
Second day: Two turtledoves and another partridge.
Third day: Three French hens, two turtledoves, and a partridge.
And so on.

For each $i = 1, \ldots, 12$, let g_i be the number of gifts sent on the ith day. Then $g_1 = 1$, and for $i = 2, \ldots, 12$ we have

$$g_i = g_{i-1} + i.$$

Now let t_n be the total number of gifts sent during the first n days of Christmas. Find a formula for t_n in the form

$$t_n = \frac{n(n+a)(n+b)}{c},$$

where $a, b, c \in \mathbb{N}$.☆

10.28 Define the binomial coefficient $\binom{n}{r}$ by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{for } r = 0, 1, 2, \ldots, n.$$

(a) Show that

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \quad \text{for } r = 0, 1, 2, \ldots, n.$$

*(b) Use part (a) and mathematical induction to prove the **binomial theorem**:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n$$

$$= a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

10.29 Prove Theorem 10.6 by using the well-ordering property of $\mathbb{N}$ instead of the principle of mathematical induction. ☆

10.30 Use the principle of mathematical induction to prove the well-ordering property of $\mathbb{N}$. Thus we could have taken Theorem 10.2 as an axiom and derived 10.1 as a theorem.

Exercise 10.31 illustrates how the basic properties of addition of natural numbers can be derived from a few simple axioms. These axioms are called the Peano axioms in honor of the Italian mathematician Giuseppe Peano, who developed this approach in the late nineteenth century. We suppose that there exist a set P whose elements are called **natural numbers** and a relation of **successor** with the following properties:

P1. There exists a natural number, denoted by 1, that is not the successor of any other natural number.

P2. Every natural number has a unique successor. If $m \in P$, then we let m' denote the successor of m.

P3. Every natural number except 1 is the successor of exactly one natural number.

P4. If M is a set of natural numbers such that
(i) $1 \in M$ and
(ii) for each $k \in P$, if $k \in M$, then $k' \in M$,
then $M = P$.

Axioms P1 to P3 express the intuitive notion that 1 is the first natural number and that we can progress through the natural numbers in succession one at a time. Axiom P4 is the equivalent of the principle of mathematical induction. Using these axioms, we can define what addition means. We begin by defining what it means to add 1.

D1. For every $n \in P$, define $n + 1 = n'$.

That is, $n+1$ is the unique successor of n whose existence is guaranteed by axiom P2. Following this pattern, it is clear that we want to define $n+2 = (n+1)+1$, $n+3 = [(n+1)+1]+1$, and so on. To define $n+m$ for all $m \in P$, we use a recursive definition:

D2. Let $n, m \in P$. If $m = k'$ and $n + k$ is defined, then define $n + m$ to be $(n+k)'$.

That is, $n + k' = (n+k)'$ or, equivalently, $n + (k+1) = (n+k) + 1$. Note that if $m \neq 1$, then the existence of k is assured by axiom P3. Now for the exercise. ☆

10.31 (a) Prove that $n + m$ is defined for all $n, m \in P$.
(b) Prove that $n + 1 = 1 + n$ for all $n \in P$.
(c) Prove that $m' + n = (m + n)'$ for all $m, n \in P$.
(d) Prove that addition is commutative. That is, prove that $n + m = m + n$ for all $m, n \in P$.
(e) Prove that addition is associative. That is, prove that $(m + n) + p = m + (n + p)$ for all $m, n, p \in P$.

For the sake of completeness, we indicate how multiplication can be defined using the Peano axioms. As you would expect by now, the definition is recursive.

D3. For every $n \in P$, define $n \times 1 = n$.
D4. Let $m, n \in P$. If $m = k'$ and $n \times k$ is defined, then define $n \times m$ to be $(n \times k) + n$. That is, $n \times (k + 1) = (n \times k) + n$.

The reader is invited to prove some of the basic properties of multiplication using D3 and D4. For a complete discussion of the development of $\mathbb{N}$ from the Peano axioms, see Henkin and others (1962).

Section 11 ORDERED FIELDS

The set $\mathbb{R}$ of real numbers can be described as a "complete ordered field." In this section we present the axioms of an ordered field and in the next section we give the completeness axiom. The purpose of this development is to identify the basic properties that characterize the real numbers. After stating the axioms of an ordered field, we derive some of the basic algebraic properties that the reader no doubt has used for years without question. It is not our intent to derive *all* these properties, but simply to illustrate how this might be done by giving a few examples. Other properties are left for the reader to prove as exercises. Having done this, we shall subsequently assume familiarity with all the basic algebraic properties (whether we have proved them specifically or not).

We begin by assuming the existence of a set $\mathbb{R}$, called the set of real numbers, and two operations $+$ and $\cdot$, called addition and multiplication, such that the following properties apply:

A1. For all $x, y \in \mathbb{R}$, $x + y \in \mathbb{R}$ and if $x = w$ and $y = z$, then $x + y = w + z$.
A2. For all $x, y \in \mathbb{R}$, $x + y = y + x$.
A3. For all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
A4. There is a unique real number 0 such that $x + 0 = x$, for all $x \in \mathbb{R}$.

A5. For each $x \in \mathbb{R}$ there is a unique real number $-x$ such that $x + (-x) = 0$.
M1. For all $x, y \in \mathbb{R}$, $x \cdot y \in \mathbb{R}$, and if $x = w$ and $y = z$, then $x \cdot y = w \cdot z$.
M2. For all $x, y \in \mathbb{R}$, $x \cdot y = y \cdot x$.
M3. For all $x, y, z \in \mathbb{R}$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
M4. There is a unique real number 1 such that $1 \neq 0$ and $x \cdot 1 = x$ for all $x \in \mathbb{R}$.
M5. For each $x \in \mathbb{R}$ with $x \neq 0$, there is a unique real number $1/x$ such that $x \cdot (1/x) = 1$. We also write x^{-1} in place of $1/x$.
DL. For all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

These first 11 axioms are called the field axioms because they describe a system known as a **field** in the study of abstract algebra. Axioms A2 and M2 are called the **commutative laws** and axioms A3 and M3 are the **associative laws**. Axiom DL is the **distributive law** that shows how addition and multiplication relate to each other. Because of axioms A1 and M1, we can think of addition and multiplication as functions that map $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. When writing multiplication we often omit the raised dot and write xy instead of $x \cdot y$.

In addition to the field axioms, the real numbers also satisfy four order axioms. These axioms identify the properties of the relation "$<$". As is common practice, we may write $y > x$ instead of $x < y$, and $x \le y$ is an abbreviation for "$x < y$ or $x = y$." The notation "$\ge$" is defined analogously. A real number x is called nonnegative if $x \ge 0$ and positive if $x > 0$. A pair of simultaneous inequalities such as "$x < y$ and $y < z$" is often written in the shorter form "$x < y < z$."

The relation "$<$" satisfies the following properties:

O1. For all $x, y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, or $x < y$ holds (trichotomy law).
O2. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.
O3. For all $x, y, z \in \mathbb{R}$, if $x < y$, then $x + z < y + z$.
O4. For all $x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $xz < yz$.

To illustrate how the axioms may be used to derive familiar algebraic properties, we include the following:

11.1 THEOREM Let x, y, and z be real numbers.

(a) If $x + z = y + z$, then $x = y$.
(b) $x \cdot 0 = 0$.
(c) $(-1) \cdot x = -x$.
(d) $xy = 0$ iff $x = 0$ or $y = 0$.
(e) $x < y$ iff $-y < -x$.
(f) If $x < y$ and $z < 0$, then $xz > yz$.

Proof: (a) If $x+z=y+z$, then

$$\begin{aligned} (x+z)+(-z) &= (y+z)+(-z) && \text{by A5 and A1,} \\ x+[z+(-z)] &= y+[z+(-z)] && \text{by A3,} \\ x+0 &= y+0 && \text{by A5,} \\ x &= y && \text{by A4.} \end{aligned}$$

(b) For any $x \in \mathbb{R}$ we have

$$\begin{aligned} x\cdot 0 &= x\cdot(0+0) && \text{by A4,} \\ x\cdot 0 &= x\cdot 0+x\cdot 0 && \text{by DL,} \\ 0+x\cdot 0 &= x\cdot 0+x\cdot 0 && \text{by A4 and A2,} \\ 0 &= x\cdot 0 && \text{by part (a).} \end{aligned}$$

(c) For any $x \in \mathbb{R}$ we have

$$\begin{aligned} x+(-1)\cdot x &= x+x\cdot(-1) && \text{by M2,} \\ &= x\cdot 1+x\cdot(-1) && \text{by M4,} \\ &= x\cdot[1+(-1)] && \text{by DL,} \\ &= x\cdot 0 && \text{by A5,} \\ &= 0 && \text{by part (b).} \end{aligned}$$

Thus $(-1)\cdot x = -x$ by the uniqueness of $-x$ in A5.

(d) See Practice 11.2.

(e) Suppose that $x<y$. Then

$$\begin{aligned} x+[(-x)+(-y)] &< y+[(-x)+(-y)] && \text{by O3,} \\ x+[(-x)+(-y)] &< y+[(-y)+(-x)] && \text{by A2,} \\ [x+(-x)]+(-y) &< [y+(-y)]+(-x) && \text{by A3,} \\ 0+(-y) &< 0+(-x) && \text{by A5,} \\ -y &< -x && \text{by A2 and A4.} \end{aligned}$$

The converse is similar.

(f) See Practice 11.3. ♦

11.2 PRACTICE Fill in the blanks in the following proof of Theorem 11.1(d).

Theorem: $xy=0$ iff $x=0$ or $y=0$.

Proof: If $x=0$ or $y=0$, then $xy=0$ by 11.1(b) and M2. Conversely, suppose that $xy=0$ and $x \neq 0$. By tautology 3.12(p), it suffices to show that $y=0$. Since $x \neq 0$, $1/x$ exists by (a) ____________. Thus

$$0 = \frac{1}{x} \cdot 0 \qquad \text{by (b) \_\_\_\_\_\_\_\_\_\_\_\_,}$$

$$0 = \frac{1}{x} \cdot (xy) \qquad \text{since } xy = 0,$$

$$0 = \left(\frac{1}{x} \cdot x\right) y \qquad \text{by (c) \_\_\_\_\_\_\_\_\_\_\_\_,}$$

$$0 = \left(x \cdot \frac{1}{x}\right) y \qquad \text{by (d) \_\_\_\_\_\_\_\_\_\_\_\_,}$$

$$0 = 1 \cdot y \qquad \text{by (e) \_\_\_\_\_\_\_\_\_\_\_\_,}$$

$$0 = y \qquad \text{by (f) \_\_\_\_\_\_\_\_\_\_\_\_.} \; \blacklozenge$$

11.3 PRACTICE Fill in the blanks in the following proof of Theorem 11.1(f).

Theorem: If $x < y$ and $z < 0$, then $xz > yz$.

Proof: If $x < y$ and $z < 0$, then $-z > 0$ by 11.1(e). Thus $x(-z) < y(-z)$ by (a) ____________. But

$$\begin{aligned} x(-z) &= x[(-1)(z)] && \text{by (b) \_\_\_\_\_\_\_\_\_\_\_\_,} \\ &= [x(-1)]z && \text{by (c) \_\_\_\_\_\_\_\_\_\_\_\_,} \\ &= [(-1)(x)]z && \text{by (d) \_\_\_\_\_\_\_\_\_\_\_\_,} \\ &= (-1)(xz) && \text{by (e) \_\_\_\_\_\_\_\_\_\_\_\_,} \\ &= -xz && \text{by (f) \_\_\_\_\_\_\_\_\_\_\_\_.} \end{aligned}$$

Similarly, $y(-z) = -yz$. Thus $-xz < -yz$. But then $yz < xz$ by (g) ____________. ♦

We have listed the field axioms and the order axioms as properties of the real numbers. But in fact they are of interest in their own right. Any mathematical system that satisfies these 15 axioms is called an **ordered field**. Thus the real numbers are an example of an ordered field. But there are other examples as well. In particular, the rational numbers $\mathbb{Q}$ are also an ordered field. Recall that

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0 \right\},$$

where $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$. Since the rational numbers are a subset of the reals, the commutative and associative laws and the order axioms are automatically satisfied.† Since 0 and 1 are rational numbers, axioms A4 and

† We have not proved that $\mathbb{Q} \subseteq \mathbb{R}$, but this relationship should come as no surprise to the reader. A rigorous proof may be found in Stewart and Tall (1977).

M4 apply. Since $-(m/n) = (-m)/n$ and $(m/n)^{-1} = n/m$, axioms A5 and M5 hold. It remains to show that the sum and product of two rationals are also rational.

11.4 PRACTICE Let a/b and c/d be rational numbers with $a, b, c, d \in \mathbb{Z}$. Show that

$$\frac{a}{b}+\frac{c}{d} \quad \text{and} \quad \frac{a}{b}\cdot\frac{c}{d}$$

are rational.

11.5 EXAMPLE For a more unusual example of an ordered field, let $\mathbb{F}$ be the set of all rational functions. That is, $\mathbb{F}$ is the set of all quotients of polynomials. A typical element of $\mathbb{F}$ looks like

$$\frac{a_n x^n + \cdots + a_1 x + a_0}{b_k x^k + \cdots + b_1 x + b_0},$$

where the coefficients are real numbers and $b_k \neq 0$. Using the usual rules for adding, subtracting, multiplying, and dividing polynomials, it is not difficult to verify that $\mathbb{F}$ is a field.

We can define an order on $\mathbb{F}$ by saying that a quotient such as above is positive iff a_n and b_k have the same sign; that is, $a_n \cdot b_k > 0$. For example,

$$\frac{3x^2+4x-1}{7x^5+5} > 0,$$

since $3 \cdot 7 > 0$. If p/q and f/g are rational functions, then we say that

$$\frac{p}{q} > \frac{f}{g} \quad \text{iff} \quad \frac{p}{q} - \frac{f}{g} > 0.$$

That is,

$$\frac{p}{q} > \frac{f}{g} \quad \text{iff} \quad \frac{pg - fq}{qg} > 0.$$

The verification that "$>$" satisfies the order axioms is left to the reader (Exercise 11.11). It turns out that the ordered field $\mathbb{F}$ has a number of interesting properties, as we shall see later.

11.6 PRACTICE Consider the field $\mathbb{F}$ of rational functions defined in Example 11.5.

(a) Which is larger, x^2 or $3/x$?

(b) Which is larger, $x/(x+2)$ or $x/(x+1)$?

There is one more algebraic property of the real numbers to which we give special attention because of its frequent use in proofs in analysis, and because it may not be familiar to the reader.

11.7 THEOREM Let $x, y \in \mathbb{R}$ such that $x \leq y + \varepsilon$ for every $\varepsilon > 0$. Then $x \leq y$.

Proof: We shall establish the contrapositive. By axiom O1, the negation of $x \leq y$ is $x > y$. Thus we suppose that $x > y$ and we must show that there exists an $\varepsilon > 0$ such that $x > y + \varepsilon$. Let $\varepsilon = (x - y)/2$. Since $x > y$, $\varepsilon > 0$. Furthermore,

$$y + \varepsilon = y + \frac{x - y}{2} = \frac{x + y}{2} < \frac{x + x}{2} = x,$$

as required. ♦

Many of the proofs in analysis involve manipulating inequalities, and one useful tool in working with inequalities is the concept of absolute value. The definition of absolute value was mentioned in Section 4, but we repeat it here for reference.

11.8 DEFINITION If $x \in \mathbb{R}$, then the **absolute value** of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The basic properties of absolute value are summarized in the following theorem.

11.9 THEOREM Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then

(a) $|x| \geq 0$,
(b) $|x| \leq a$ iff $-a \leq x \leq a$,
(c) $|xy| = |x| \cdot |y|$,
(d) $|x + y| \leq |x| + |y|$.

Proof: (a) There are two cases. If $x \geq 0$, then $|x| = x \geq 0$. On the other hand, if $x < 0$, then $|x| = -x > 0$. In both cases $|x| \geq 0$.

(b) Since $x = |x|$ or $x = -|x|$, it follows that $-|x| \leq x \leq |x|$. Now if $|x| \leq a$, then we have

$$-a \leq -|x| \leq x \leq |x| \leq a.$$

Conversely, suppose that $-a \leq x \leq a$. If $x \geq 0$, then $|x| = x \leq a$. And if $x < 0$, then $|x| = -x \leq a$. In both cases, $|x| \leq a$.

(c) See Exercise 11.5.

(d) As in part (b), we have

$$-|x| \le x \le |x| \quad \text{and} \quad -|y| \le y \le |y|.$$

Adding the inequalities together, we obtain

$$-(|x|+|y|) \le x+y \le |x|+|y|,$$

which implies that $|x+y| \le |x|+|y|$ by part (b). ♦

Part (d) of Theorem 11.9 is referred to as the **triangle inequality**:

$$|x+y| \le |x|+|y|.$$

It is also useful in other forms. For example, letting $x = a - c$ and $y = c - b$, we obtain

$$|a-b| \le |a-c|+|c-b|.$$

If we think of the real numbers as being points on a line, then $|a-b|$ represents the distance from a to b. Thus the distance from a to b is less than or equal to the sum of the distances from a to c and c to b. It is possible to generalize this to higher dimensions, where a, b, and c are the vertices of a triangle. It is this more general setting that gives rise to the name "triangle inequality." [See Section 15.]

Review of Key Terms in Section 11

Field axioms	Order axioms	Absolute value

ANSWERS TO PRACTICE PROBLEMS

11.2 (a) M5; (b) Theorem 11.1(b); (c) M3; (d) M2; (e) M5; (f) M2 and M4.

11.3 (a) O4; (b) Theorem 11.1(c); (c) M3; (d) M2; (e) M3; (f) Theorem 11.1(c); (g) Theorem 11.1(e).

11.4 From the usual rules of arithmetic we have

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

Since sums and products of integers are always integers,

$$\frac{ad+bc}{bd} \quad \text{and} \quad \frac{ac}{bd}$$

are both rational.

11.6 (a) $\frac{x^2}{1} - \frac{3}{x} = \frac{x^3-3}{x} > 0$, so $x^2 > \frac{3}{x}$.

(b) $\frac{x}{x+2} - \frac{x}{x+1} = \frac{-x}{x^2+3x+2} < 0$, so $\frac{x}{x+2} < \frac{x}{x+1}$.

EXERCISES

*Exercises marked with * are used in later sections and exercises marked with ☆ have hints or solutions in the back of the book.*

11.1 Mark each statement True or False. Justify each answer.

(a) Axioms A1 to A5, M1 to M5, and DL describe an algebraic system known as a field.
(b) The property that $x + y = y + x$ for all $x, y \in \mathbb{R}$ is called an associative law.
(c) If $x, y, z \in \mathbb{R}$ and $x < y$, then $xz < yz$.

11.2 Mark each statement True or False. Justify each answer.

(a) Axioms A1 to A5, M1 to M5, DL, and O1 to O4 describe an algebraic system known as an ordered field.
(b) If $x, y \in \mathbb{R}$ and $x < y + \varepsilon$ for every $\varepsilon > 0$, then $x < y$.
(c) If $x, y \in \mathbb{R}$, then $|x+y| \geq |x| + |y|$.

11.3 Let x, y, and z be real numbers. Prove the following.

(a) $-(-x) = x$. ☆
(b) $(-x) \cdot y = -(xy)$ and $(-x) \cdot (-y) = xy$.
(c) If $x \neq 0$, then $(1/x) \neq 0$ and $1/(1/x) = x$.
(d) If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.
(e) If $x \neq 0$, then $x^2 > 0$. ☆
(f) $0 < 1$. ☆
(g) If $x > 1$, then $x^2 > x$.
(h) If $0 < x < 1$, then $x^2 < 1$.
(i) If $x > 0$, then $1/x > 0$. If $x < 0$, then $1/x < 0$.
(j) If $0 < x < y$, then $0 < 1/y < 1/x$.
(k) If $xy > 0$, then either (i) $x > 0$ and $y > 0$, or (ii) $x < 0$ and $y < 0$.
(l) For each $n \in \mathbb{N}$, if $0 < x < y$, then $x^n < y^n$. ☆
(m) If $0 < x < y$, then $0 < \sqrt{x} < \sqrt{y}$. ☆

***11.4** Prove: If $x \geq 0$ and $x \leq \varepsilon$ for all $\varepsilon > 0$, then $x = 0$.

11.5 Prove Theorem 11.9(c): $|xy| = |x| \cdot |y|$. ☆

***11.6** (a) Prove: $\big||x| - |y|\big| \leq |x - y|$.
(b) Prove: If $|x - y| < c$, then $|x| < |y| + c$.
(c) Prove: If $|x - y| < \varepsilon$ for all $\varepsilon > 0$, then $x = y$.

***11.7** Suppose that $x_1, x_2, \ldots, x_n$ are real numbers. Prove that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|. \text{ ☆}$$

11.8 Let $P = \{x \in \mathbb{R}: x > 0\}$. Show that P satisfies the following:

(a) If $x, y \in P$, then $x + y \in P$.
(b) If $x, y \in P$, then $x \cdot y \in P$.
(c) For each $x \in \mathbb{R}$, exactly one of the following three statements is true: $x \in P$, $x = 0$, $-x \in P$.

11.9 Let F be a field and suppose that P is a subset of F that satisfies the three properties in Exercise 11.8. Define $x < y$ iff $y - x \in P$. Prove that "$<$" satisfies axioms O1, O2, and O3. Thus in defining an ordered field, either we can begin with the properties of "$<$" as in the text, or we can begin by identifying a certain subset as "positive."

11.10 Prove that in any ordered field F, $a^2 + 1 > 0$ for all $a \in F$. Conclude from this that if the equation $x^2 + 1 = 0$ has a solution in a field, then that field cannot be ordered. (Thus it is not possible to define an order relation on the set of all complex numbers that will make it an ordered field.)

11.11 Let $\mathbb{F}$ be the field of rational functions described in Example 11.5. ☆

(a) Show that the ordering given there satisfies the order axioms O1, O2, and O3.
(b) Write the following polynomials in order of increasing size:

$$x^2, \ -x^3, \ 5, \ x+2, \ 3-x.$$

(c) Write the following functions in order of increasing size:

$$\frac{x^2+2}{x-1}, \ \frac{x^2-2}{x+1}, \ \frac{x+1}{x^2-2}, \ \frac{x+2}{x^2-1}.$$

11.12 Let $S = \{a, b\}$ and define two operations $\oplus$ and $\otimes$ on S by the following charts:

$\oplus$	a	b
a	a	b
b	b	a

$\otimes$	a	b
a	a	a
b	a	b

(a) Verify that S together with $\oplus$ and $\otimes$ satisfy the axioms of a field.
(b) Identify the elements of S that are "0," "1," and "−1."

11.13 To actually construct the rationals $\mathbb{Q}$ from the integers $\mathbb{Z}$, let $S = \{(a, b): a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. Define an equivalence relation "$\sim$" on S by $(a, b) \sim (c, d)$ iff $ad = bc$. We then define the set $\mathbb{Q}$ of rational numbers to be the set of equivalence classes corresponding to $\sim$. The equivalence class determined by the ordered pair (a, b) we denote by $[a/b]$. Then

$[a/b]$ is what we usually think of as the fraction a/b. For $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$, we define addition and multiplication in $\mathbb{Q}$ by

$$[a/b]+[c/d] = [(ad+bc)/bd],$$
$$[a/b]\cdot[c/d] = [(ac)/(bd)].$$

We say that $[a/b]$ is *positive* if $ab \in \mathbb{N}$. Since $a, b \in \mathbb{Z}$ with $b \neq 0$, this is equivalent to requiring $ab > 0$. The set of positive rationals is denoted by $\mathbb{Q}^+$, and we define an order "$<$" on $\mathbb{Q}$ by

$$x < y \quad \text{iff} \quad y - x \in \mathbb{Q}^+.$$

(a) Verify that $\sim$ is an equivalence relation on S.
(b) Show that addition and multiplication are well-defined. That is, suppose $[a/b] = [p/q]$ and $[c/d] = [r/s]$. Show that $[(ad + bc)/bd] = [(ps + qr)/qs]$ and $[ac/bd] = [pr/qs]$.
(c) For any $b \in \mathbb{Z} \setminus \{0\}$, show that $[0/b] = [0/1]$ and $[b/b] = [1/1]$.
(d) For any $a, b \in \mathbb{Z}$ with $b \neq 0$, show that $[a/b] + [0/1] = [a/b]$ and $[a/b]\cdot[1/1] = [a/b]$. Thus $[0/1]$ corresponds to zero and $[1/1]$ corresponds to 1.
(e) For any $a, b \in \mathbb{Z}$ with $b \neq 0$, show that $[a/b] + [(-a)/b] = [0/1]$ and $[a/b]\cdot[b/a] = [1/1]$.
(f) Verify that the set $\mathbb{Q}$ with addition, multiplication, and order as given above satisfies the axioms of an ordered field.

11.14 Construct the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$ in a method similar to that used in Exercise 11.13 by defining an appropriate equivalence relation on $\mathbb{N} \times \mathbb{N}$.

Section 12 THE COMPLETENESS AXIOM

In the preceding section we presented the field and order axioms of the real numbers. Although these axioms are certainly basic to the real numbers, by themselves they do not characterize $\mathbb{R}$. That is, we have seen that there are other mathematical systems that also satisfy these 15 axioms. In particular, the set $\mathbb{Q}$ of rational numbers is an ordered field. The one additional axiom that distinguishes $\mathbb{R}$ from $\mathbb{Q}$ (and from other ordered fields) is called the completeness axiom. Before presenting this axiom, let us look briefly at why it is needed—at why the rational numbers by themselves are inadequate for analysis.