Midterm Exam
Math 117
Winter 2013
Prof. R. Ye

Your Name:
Your Signature:
Your Perm Number:

Scores:
1.
2.
3.
4.
Total: (out of 100)

Please present detailed steps of your solutions.

1. (25 points) For each $n \in \mathbb{N}$, Let $P(n)$ be a statement concerning n which is either true or false. Assume the following two conditions:
1) $P(1)$ and $P(2)$ are true.
2) For each $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n+2)$ is also true.
Prove that $P(n)$ is true for all $n \in \mathbb{N}$. Hint: You can use the well-ordering axiom, the principle of mathematical induction, or Peano axioms.

Proof 1 Define $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$. We claim that $S = \emptyset$. Assume that S is nonempty. By the well-ordering axiom of natural numbers, there is a smallest number $n_0 \in S$. Since $P(1)$ and $P(2)$ are true, there holds $n_0 \geq 3$. Then $n_0 - 2 \in \mathbb{N}$. Since $n_0 - 2 < n_0$, there holds $n_0 - 2 \notin S$, otherwise n_0 would not be the smallest number in S. Hence $P(n_0 - 2)$ is true. By the assumption 2) we infer that $P(n_0)$ is true, contradicting the fact that $n_0 \in S$. We conclude that S is empty, and hence $P(n)$ is true for all $n \in \mathbb{N}$. ■

Proof 2 Set $Q(n) = P(2n)$ for $n \in \mathbb{N}$. Then $Q(1) = P(2)$ is true. If $Q(n) = P(2n)$ is true, then $Q(n+1) = P(2n+2)$ is also true by the assumption 2). By the principle of mathematical induction, $Q(n) = P(2n)$ is true for all $n \in \mathbb{N}$, i. e. $P(n)$ is true for all even natural numbers n.

Next set $H(n) = P(2n-1)$. Then $H(1) = P(1)$ is true. If $H(n) = P(2n-1)$ is true, then $H(n+1) = P((2n-1)+2)$ is true by the assumption 2). By the principle of mathematical induction we infer that $H(n) = P(2n-1)$ is true for all $n \in \mathbb{N}$. This means that $P(n)$ is true for all odd natural numbers n.

Combining the above two conclusions we infer that $P(n)$ is true for all $n \in \mathbb{N}$. ■

2. 25 points) 1) Assume $0 < x < y$. Show $0 < 1/y < 1/x$..
2) Assume $x^2 = y^2$. Determine the relation between x and y. (A detailed proof is required.)

Proof. 1) First we show that $1/x > 0$. Assume $0 > 1/x$. (As the inverse of x, $1/x \neq 0$.) Since x is positive, we can multiply this inequality by x to deduce $0 \cdot x > 1/x \cdot x$. Hence $0 > 1$. This contradicts the fact that $1 > 0$, which has been established in the lectures. It follows that $1/x > 0$, Similarly, we have $1/y > 0$.

(The proof for $1 > 0$ is as follows. Assume $0 > 1$. Adding -1 to both sides of this inequality we deduce $-1 > 0$. Multiplying with -1 we infer $(-1)^2 > 0$. Hence $1 > 0$, contradicting the assumption $0 > 1$. Here, the fact that $(-1)^2 = 1$ can be seen as follows. We have $1 + (-1) = 0$. Hence $(-1) + (-1)^2 = 0$. Adding 1 to both sides we infer that $(-1)^2 = 1$.)

Now we multiply the inequality $x < y$ by $1/x$ to deduce $1 < y/x$. Then we multiply by $1/y$ to arrive at $1/y < 1/x$. ∎

2) Assume $x^2 = y^2$. Then $x^2 - y^2 = 0$, and hence $(x + y)(x - y) = 0$. It follows that $x + y = 0$ or $x - y = 0$. Conversely, if $x + y = 0$ or $x - y = 0$, then $(x + y)(x - y) = 0$, and hence $x^2 - y^2 = 0$, i.e. $x^2 = y^2$. We conclude that the relation between x and y is $x = y$ or $x = -y$. In other words, the relation is $|x| = |y|$. ∎

An Alternative Argument:

Assume $x \neq 0$. Then $y \neq 0$. (Otherwise $x^2 = 0^2 = 0$, and hence $x = 0$.) We first claim $|x| \leq |y|$. Assume $|x| > |y|$. Multiplying this inequality with $|x|$ we infer $|x|^2 > |x||y|$. Multiplying the same inequality with $|y|$ we deduce $|x||y| > |y|^2$. It follows that $|x|^2 > |y|^2$, which means $x^2 > y^2$, contradicting the assumption $x^2 = y^2$. Hence we conclude that $|x| \leq |y|$. In a similar way we can prove the claim $|x| \geq |y|$. Hence we infer that $|x| = |y|$.

If $x = 0$, then $y^2 = x^2 = 0$ and hence $y = 0$. Hence we also have $|x| = |y|$.

Conversely, if $|x| = |y|$, then we infer $|x|^2 = |y|^2$, which means $x^2 = y^2$.

In conclusion, the equality $x^2 = y^2$ is equivalent to the equation $|x| = |y|$, which is equivalent to the following relation: $x = y$ or $x = -y$. ∎

3. (25 points) Let $x \in \mathbb{R}$. Prove that $x = \sup\{q \in \mathbb{Q} : q < x\}$.

Proof 1) Define $S = \{q \in Q : q < x\}$. It is bounded above because x is an upper bound for it. By the Archimedean property, we can find a natural number $n > -x$. Then $-n < x$, and hence $-n \in S$. Thus S is nonempty. By the completeness axiom, $y = \sup S$ exist.
2) Since x is an upper bound for S and y is the least upper bound for S, there holds $y \leq x$.
3) We claim that $y = x$. Assume $y \neq x$. By 2) we then have $y < x$. By the density of $\mathbb{Q}$, there exists a number $r \in \mathbb{Q}$ such that $y < r < x$. Then $r \in S$, and hence $r \leq y$, contradicting the fact that $y < r$. ■

4. (25 points) Let $S = \{\frac{1}{n} + \frac{1}{m} - \frac{1}{k}$ for all $n, m, k \in \mathbb{N}\}$.
1) Find $\sup S$ and $\inf S$.
2) Does $\max S$ exist? Why?
3) Does $\min S$ exist? Why?

Proof
1) For a fixed k, the maximum of $\frac{1}{n} + \frac{1}{m} - \frac{1}{k}$ is $1 + 1 - \frac{1}{k} = 2 - \frac{1}{k}$. Letting k become bigger and bigger we see that $\sup S = 1+1-0 = 2$. To get the infimum, we take $k = 1$ and let n and m become bigger and bigger. It follows that $\inf S = 0 + 0 - 1 = -1$.
2) $\max S$ does not exist. Assume that it exists. Then $\max S = \sup S = 2$. But $\max S = \frac{1}{n} + \frac{1}{m} - \frac{1}{k}$ for some n, m and k. It follows that $2 = \max S \leq 1 + 1 - \frac{1}{k} < 2$. This is a contradiction.
3) $\min S$ does not exist. Assume that it exists. Then $\min S = \inf S = -1$, But $\min S = \frac{1}{n} + \frac{1}{m} - \frac{1}{k}$ for some n, m and k. Hence $\min S \geq \frac{1}{n} + \frac{1}{m} - 1 > -1$. This is a contradiction. ■