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The classification of topological four manifolds with infinite cyclic fundamental group

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The Classification of topological four manifolds with infinite cyclic fundamental group

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics

by

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1993

The dissertation of Zhenghan Wang is approved, and it is acceptable in quality and form for publication on microfilm:

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1993

To my parents, who gave me life.

To my teachers, who changed my life.

Wu Wei: Let Nature take its course ...

Lao-tzu

Contents

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Abstract of the Dissertation

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In this thesis, topological four manifolds with $\pi_1 = Z$ are classified. The method is first used to give a new proof of the classification theorem of M. Freedman for the $\pi_1 = 0$ case.

In chapter 2 and 3, we supply details for some background materials. In chapter 4, the complete classification is obtained. The method we used is a mixture of the algebraic approach of M. Kreck and the geometric approach of Freedman and Quinn. First the stable classification is obtained using the work of M. Kreck. Then some properties of 4-manifolds with $\pi_1 = Z$ are derived.Finally, the classification is completed by an explicit construction for existence, and by a sum decomposition theorem for uniqueness. Part of the theorem in the orientable case has been obtained by Freedman-Quinn and M. Kreck already. Our method is different and the result is complete. The classification theorem in nonorientable case is new. In chapter 5, we study the classification of self homeomorphisms of a 4-manifold with $\pi_1 = Z$ up to pseudoisotopy, and some immediate corollaries are drawn together with a discussion for an open problem in chapter 6.

The classification theorem is an analogue of the 1-connected case: the homeomorphism type of a 4-manifold with $\pi_1 = Z$ is determined by ω_1, the intersection from on π_2 and the Kirby-Siebenmann invariant. Our method works for the general free fundamental group case if the disk theorem is true.

Chapter 1

Introduction

Our knowledge of topological four manifolds has been changed dramatically since the breakthrough of Michael H. Freedman in 1981. He proved the Poincare conjecture for topological four manifolds by and obtained a complete classification of the simply connected four manifolds (a technical assumption in the original classification was later removed by F. Quinn). His classification follows the surgery program. It says that the homeomorphism type of a simply connected closed 4-manifolds is determined by the intersection form and the Kirby-Siebenmann invariant. As a corollary, the empty intersection form is the 4-sphere. Later, a new proof appeared in the book of Freedman-Quinn, *Topology of 4-manifolds.* The new proof has the advantage that it obtained the classification directly without solving the difficult problem of determining the homotopy type beforehand. After the work of Freedman, Prof. M. Kreck developed a theory which generalizes the classical surgery. They classify four manifolds by first obtaining the stable classification, i.e. determining the homeomorphism type up to connected sum with copies of $S^2 \times S^2$, and then obtain the classification by cancellation using Freedman's disk embedding theorem.

In this way, Prof. M. Kreck and I. Hambleton obtained the complete classification for orientable four manifolds with finite cyclic fundamental group. The nonorientable case is only obtained for fundamental group Z_2 by M.Kreck, I. Hambleton and P. Teichner recently. The algebra involved in the classification is very complicated. It seems very difficult to achieve some other complete classification. In this direction, for finite fundamental groups, a lot of partial results are obtained by P. Teichner in his thesis. For the fundamental group Z case, the classification for the orientable case is stated in chapter 10 of the book of Freedman-Quinn without the details of proof. Actually the uniqueness part of the classification is not completely correct. It is the problem to complete the classification that starts this work. Part of the classification theorem for the orientable case was also obtained by M. Kreck using a totally different method. Our method is different from both of them. For the nonorientable case, we combine the stable classification theory of M.Kreck with the geometric method of Freedman-Quinn. It is interesting to see that the fundamental group Z case is really much easier than the finite fundamental group case in general.

The obvious invariants for the classification is π_1, ω_1, π_2, the intersection form on π_2 and the Kirby-Siebenmann invariant. The classification is modeled on the classification theorem of Freedman for the simply connected case. The classification theorem says these invariants determine the the homeomorphism type of the four manifolds with fundamental group Z.

Suppose M is a closed 4-manifold with $\pi_1 = J$ (to distinguish the infinite cyclic group from the coefficient ring Z, we will denote the infinite cyclic group by J), then $H_2(M; Z[J]) = \pi_2 M$ is a free $Z[J]$-module. When M is orientable, then the intersection form is a nonsingular hermitian form with respect to the group ring involution on this free module. We have the following:

Theorem 1 *(1).* **Existence** *: Suppose (H, λ) is a nonsingular hermitian form on a finitely generated free $Z[J]$-module, $k \in Z_2$ and if λ is even, then we assume $k = (signature \lambda)/8, mod\ 2$, then there exists an oriented closed 4-manifold with $\pi_1 = J$, intersection form λ and Kirby-Siebenmann invariant k.*

(2). **Uniqueness***: Suppose M, N are closed oriented 4-manifolds with $\pi_1 = J$, $h : H_2(M; Z[J]) \longrightarrow H_2(N; Z[J])$ is a $Z[J]$-isomorphism and ks(M)=ks(N). Then there exists a homeomorphism $f : M \longrightarrow N$ which induces the given identification of fundamental groups, preserving orientation and such that $f_* = h$. f is not unique up to pseudoisotopy.*

The proof of the existence (1) is almost explicit. Given the data, we construct a 4-manifold by attaching 2-handles on a link in the solid torus. The uniqueness is proved using the celebrated disk theorem of Freedman.

The classification theorem of the nonorientable case is analogous. In this case, hermitian form is replaced by ω_1-hermitian form, where ω_1 is the first Stiefel-Whitney class. But the proof is quite different. The existence is not explicit. The manifold is constructed as follows: first for each form, we realize it up to stabilization of hyperbolic forms, then realize the original form by cancellation of hyperbolic forms. Uniqueness is proved using a sum decomposition theorem which is a generalization of connected sum.

Given an isometry h as in (2), we are interested in classifying the homeomorphisms that induce the same h up to pseudoisotopy. It has been shown that if M is even, then it is unique. But if it is odd, then there are at most two classes, but whether it is one or two is not clear right now.

We also use homotopy theoretic method to classify homeomorphisms because of the following:

Theorem 2 *Suppose M, N are oriented closed 4-manifolds with $\pi_1 = Z$, f, g:$M \longrightarrow N$ are two homeomorphisms, then they are homotopic iff they are pseudoisotopic.*

The theorem is proved by surgery. By computation, it is shown the obstruction from homotopy to pseudoisotopy is only a Z_2 obstruction. Then a self homotopy equivalence of $N \times I$ is constructed to carry this obstruction. Therefore, the obstruction can always be killed by choosing an appropriate homotopy.

As an application of the classifications, it is clear that the homotopy type of a 4-manifold with $\pi_1 = Z$ is determined by the intersection form on π_2. Another corollary is that some manifolds such as $S^1 \times S^3 \sharp S^2 \times S^2$ have exotic smooth structures.

Chapter 2

Background Material

2.1 Notational Conventions

Most of the notation and terminology is standard. We follow closely [FQ] for these matters whenever possible. The following ever present hypothesis is always assumed, unless otherwise clearly stated. Manifold always means topological manifold. We work in the topological category and all basic tools such as transversality, bundle etc are available by [FQ]. All surfaces are always assumed to be oriented. Surfaces are not neccessarily connected, but all components are compact (S^2 or D^2, etc.). By component of an immersed surface we mean component in the manifold sense, and so components may intersect. All intersection of arcs, surfaces, 3-disks in the ambient manifold are assumed to be generically positioned subject to the constraints imposed by the hypothesis. Surfaces and 3-disks are often denoted by A, B, C, D ... and the union of a collection such as C_i is also denoted by C if no confusion results. We are constantly moving sets like A, B, C.. etc without renaming them, to keep the notations simple. A simple closed curve in a space is often thought as representing an element in the fundamental group when the base point are appropriately chosen. The choice of a parametrization is either unimportant or clear from the context. All moves are performed in the interior of an ambient manifold.

2.2 Basic invariants

In this section, some basic invariants are discussed. The discussion is rather incomplete. For a complete treatment, see[FQ], [Ki].

2.2.1 Symmetric bilinear forms over the integers

Let M be an oriented manifold of dimension 4. Then the intersection numbers define a symmetric nonsingular bilinear form, denoted by S, on $H_2(M;Z)$. Recall that S is defined as follows: for any two classes $x, y \in H_2(M;Z)$, represent them by embeded surfaces denoted by x, y, too. Then $S(x,y)$ is the number of intersections between x, and y counted according to sign.

Given a symmetric nonsingular bilinear form S on a free Z-module X of finite dimension, there are three basic invariants: rank, signature and type. The rank is the dimension of the module X. The signature is the number of positive entries minus the number of negative entries if the form is diagonalized over the rationals. The type is even if $S(x,x) = 0 \ mod \ 2$ for all $x \in X$, and odd otherwise. The indefinite odd form is always a direct sum $\oplus p(1) \oplus q(-1)$ for some intergers p,q, and the signature of an even form is always divisible by 8[Ki].

The type, signature of a closed 4-manifold is the type, signature of its integral intersection form.

For example by convention for S^4, the intersection form is $\emptyset$, it is even. For CP^2, the intersection form is $(+1)$, which is odd and of signature 1. For $\overline{CP^2}$, the intersection form is (-1), which is odd and of signature -1. For $S^2 \times S^2$, the intersection form is the standard hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is even and of signature 0. There is a manifold with intersection form E_8 by Freedman which will be denoted by E_8, too.

2.2.2 The Kirby-Siebenmann invariant

Let M be a compact connected topological 4-manifold. Then there is a unique obstruction $ks(M) \in H^4(M; Z_2)$ to put a smooth structure on $M \sharp k(S^2 \times S^2)$ for some $k \geq 0$. It is an unoriented bordism invariant i.e if $M \cup N = \partial W$, then ks(M)=ks(N). It is additive under connected sum.

For example, there is the famous Chern manifold $*CP^2$ by Freedman which is homotopic to CP^2 but has ks=1.

2.2.3 Rochlin's theorem

One early success in smooth 4-manifold theory is the Rochlin's theorem. It relates the Kirby-Siebenmann invariant with the signature of a spin 4-manifold. For a closed topological spin 4-manifold M, it says $\frac{sign(M)}{8} \cong ksM \ mod\ 2$. For a smooth closed spin 4-manifold M, ksM=0, therefore, the signature is divisible by 16. Recall the divisibility by 8 for the signature of even forms. In the following, ks always denotes the Kirby-Siebenmann invariant.

2.2.4 Intersection and self intersection numbers

For 1-connected 4-manifolds, the integral intersection form classifies the manifolds up to ks. But for nonsimply connected manifold, this is far from enough. The obvious invarians are the fundamental group, the second homotopy group and the intersection form on π_2. The group π_2 will be treated as a $Z[\pi_1]$-module or an abelian group according to the context.

Given a closed 4-manifold M, a base point $* \in M$. Let $\tilde{M}$ be the universal covering and $\tilde{*}$ be a point fixed over $*$ in $\tilde{M}$. Denote $Z[\pi_1]$ by Λ. The following identification is well known: $\pi_2(M) = \pi_2(\tilde{M}) = H_2(\tilde{M}; Z) = H_2(M; \Lambda)$ as a Λ-module. The module $H_2(\tilde{M}, Z)$ acquires the right Λ-module structure via the covering transformations.

Definition 1 *Let $S : H_2(\tilde{M}, Z) \times H_2(\tilde{M}, Z) \longrightarrow Z$ be the usual integral intersection form of homology classes. It is easy to check that S is π_1-equivariant i.e for any $x, y \in H_2(\tilde{M}; Z)$ and $g \in \pi_1$, $S(xg, yg) = S(x, y)$. Now define $\lambda : \pi_2(M) \times \pi_2(M) \longrightarrow \Lambda$ as follows: for any $x, y \in \pi_2(M)$, $\lambda(x, y) = \sum_{g \in \pi_1 M} S(x, yg^{-1})g$. By compactness, this is a finite sum.*

Let $^-$ be the involution on Λ given by $\overline{\sum_g \lambda_g g} = \sum_g \omega_1(g)\lambda_g g^{-1}$, where $\omega_1 : \pi_1 \longrightarrow Z_2$ is the first Stiefel-Whitney class. The following properties can be deduced directly:

Proposition 1 *(1) λ is Z-linear*

(2) for any $\tau \in \Lambda$, $\lambda(x, y\tau) = \lambda(x, y)\tau$, i.e. for any $x \in \pi_2 M$, $y \longrightarrow \lambda(x, y)$ is a Λ-homomorphism from $\pi_2 M \longrightarrow \Lambda$.

(3) $\lambda(y, x) = \overline{\lambda(x, y)}$.

From (1) to (3), it follows that λ is ω_1-hermitian. The form λ is called *nonsingular* if the homomorphism $\pi_2 M \longrightarrow Hom_\Lambda(\pi_2 M, \Lambda)$ is an isomorphism.

Let ω_2 be the second Stiefel-Whitney class of the stable normal bundle. Then ω_2 defines a map $H_2(M; Z_2) \longrightarrow Z_2$ by the mod 2 intersection numbers, i.e. for any $x \in H_2(M; Z_2)$, $\omega_2(x) = x \cdot x \ mod\ 2$, where $x \cdot x \ mod\ 2$ is the mod 2 intersection number. By following $\pi_2 M \longrightarrow H_2(M; Z) \longrightarrow H_2(M; Z_2) \longrightarrow Z_2$, ω_2 defines a map $\pi_2 M \longrightarrow Z_2$, i.e given a class $x \in \pi_2(M)$, $\omega_2(x)$ is the value of ω_2 on the image of x first mapped to $H_2(M, Z)$ by the Hurewicz's map, then to $H_2(M, Z_2)$ by reduction modulo 2.

If $x \in H_2(M,Z)$, then $\omega_2(x) = S(x,x)\ mod\ 2$, where S is the intersection form on $H_2(M,Z)$. For an oriented closed 4-manifold, it is spin iff $\omega_2 = 0$. If M is simply connected, then it is equivalent to the intersection form being even. But if M is not simply connected, this is not enough, even $\omega_2(x) = 0\ mod\ 2$ for any integral class $x \in H_2(M;Z)$. For example, let $X = S^2 \times S^2/Z_2$, where Z_2 acts on (x, y) by sending it to (-x, -y). It is easy to check $H_2(X;Z)/torsion = 0$. But the diagonal RP^2 has mod 2 intersection number 1. So $\omega_2(RP^2) = RP^2 \cdot RP^2 = 1$ which implies $\omega_2 \neq 0$. Therefore X is not spin. In the following, we will also use the following terminology:

Definition 2 *Let M be a 4-manifold, M is weakly even if* $\omega_2 : \pi_2 M \longrightarrow Z_2$ *is trivial. Otherwise, M is weakly odd.*

Recall that a manifold being even or odd is determined by the intersection form on $H_2(M;Z)$. In particular, M is even implies that M is weakly even , but the above example shows that the converse is not true.

Now we define the self intersection number for spherical classes. Given $x \in \pi_2 M$, represent x by an immersed 2-sphere, denoted also by x. Fix a path from the base point $* \in M$ to a base point on the 2-sphere x. For each transverse self intersection point, drawing an arc on x from the base point of x to the double point, jumping to the other sheet, drawing another arc from the double point to the base point of x and coming back to the base point $* \in M$ from the base point of x along the chosen path. All the choices of paths above avoid the other double points. This defines a closed loop in M and therefore, an element of $Z[\pi_1 M]$. But there is no natural choice of the two sheets.

Definition 3 *By choosing one of these loops for each double point, sum over all intersection points and divided out by the ideal* $I = \{a - \bar{a}\}$. *We define the self* *intersection number $\mu : \pi_2 M \longrightarrow \Lambda/I$. It is only well defined on $ker(\omega_2)$ or into $\Lambda/Z + I$.*

Proposition 2 *The following properties can be checked by drawing pictures:*

(4) $\lambda(x,x) = \mu(x) + \overline{\mu(x)} + \chi_N(x)$, where $\chi_N(x)$ is the Euler number of x.

(5) $\mu(x+y) = \mu(x) + \mu(y) + \lambda(x,y)$ mod I

(6) if $g \in \pi_1$, then $\mu(xg) = \bar{g}\mu(x)g$.

From $Z[\pi_1]$ to Z, there are two natural homomorphisms ϵ and ϵ_1. The map $\epsilon : Z[\pi_1] \longrightarrow Z$ is defined by $\epsilon(\sum_g \lambda_g g) = \sum \lambda_g$ and ϵ_1 is defined by $\epsilon_1(\sum_g \lambda_g) = \lambda_1$, i.e. the coefficient at the identity. For the selfintersection nunmber μ, there is an associated *reduced self intersection nubmber* $\tilde{\mu}$ defined by $\tilde{\mu} = \mu - \epsilon_1$. i.e. $\tilde{\mu} \in Z[\pi_1]/\{Z + a - \bar{a}\}$. The difference between μ and $\tilde{\mu}$ is that $\tilde{\mu}$ is an homotopy invariant as λ, but μ is not[FQ]. It is not difficult to give an immersed 2-sphere with $\tilde{\mu} = 0$ which can not be regular homotopic to an embedding. For example, the core disk of the 2-handle in the Chern manifold $*CP^2$.

There is a relation between λ on $\pi_2 M$ and S on $H_2(M, Z)$. On the image of the Hurewicz map in $H_2(M, Z)$, S is the composition of λ and ϵ_1. But λ can not determine S since not every class of H_2 is spherical, for example T^4, and λ is not determined by S, for example $X = S^2 \times S^2/Z_2$ above.

2.3 Classical surgery

One central theorem in manifold theory is the surgery sequence. In this section, we collect some facts that will be used later for convenience. In dimension 4, the new feature is that the surgery sequence is exact with the condition that the fundamental group is good[FQ].

Let (X, N) be a Poincare pair of formal dimension 4, with N a 3-manifold, then

$$L_5^s(\pi_1 X, \omega_1) \longrightarrow S_{TOP}(X, N) \longrightarrow NM_{TOP}(X, N) \longrightarrow L_4^s(\pi_1 X, \omega_1)$$

is exact if $\pi_1 X$ is good. This sequence extends to the left for $X \times I$ if X is a manifold by:

$$\longrightarrow S_{TOP}(X \times I, \partial(X \times I)) \longrightarrow NM_{TOP}(X \times I, \partial) \rightarrow$$

$$\rightarrow L_{n+1}^s(\pi_1 X, \omega_1) \longrightarrow S_{TOP}(X, N) \longrightarrow .$$

The sets for $X \times I$ have natural group structures obtained by glueing in the I coordinate. With respect to this the left part of the sequence is exact as a sequence of groups. Recall that the Wall group $L_4(0)$ for the trivial group is Z with generator the surgery problem $E_8 \longrightarrow S^4$.

Proposition 3 *(1): For the orientable case,*

$$L_k(Z^n, +) = L_k(Z^{n-1}, +) \oplus L_{k-1}(Z^{n-1}, +)$$

In particular, $L_4(0) = L_4(Z, +)$. *For the nonorientable case, it is*

$$L_4(Z, -) = Z_2, L_5(Z, -) = 0, L_6(Z, -) = Z_2, L_7(Z, -) = Z_2$$

of periodic 4. The generator of $L_4(Z, -)$ *is given by* $E_8 \otimes_Z Z[Z]$ *i.e.* $L_4(0) \longrightarrow L_4(Z, -)$ *is onto.*

(2): If $NM_{TOP}(X, N)$ *is nonempty, then*

$$NM_{TOP}(X, N) = H^4(X, N; Z) \oplus H^2(X, N; Z_2).$$

(3): Let E be a D^j *bundle over* S^1. *If* $N \longrightarrow E$ *is a homotopy equivalence which is a homeomrophism on the boundary, then it is homotopic rel boundary to a homeomorphism.*

Proof: (1): The formula was proved in [Sh1]. Geometrically, it can be described as follows: Let X^{k-1} be a Poincare space of formal dimension k-1 with $\pi_1 X = Z^{n-1}$ with a bundle ν_0. Let ν_1 be the trivial bundle over S^1. Suppose $f : M \longrightarrow X \times S^1$ is a degree 1 map and F is a framing i.e. a trivialization of $TM \oplus f^*(\nu_0 \oplus \nu_1)$ such that $(M, f, \nu_0 \oplus \nu_1, F)$ defines an element in $L_k(Z^n)$. Make f transverse to X and set $N = f^{-1}(X)$. Without loss of generality, assume $f \mid N : N \longrightarrow X$ is of degree 1 and induces an isomorphism on π_1. It induces a surgery problem $(N, f \mid N, \nu_0, F \mid N)$ which defines an element in $L_{k-1}(Z^{n-1})$. Since every element in $L_k(Z^n)$ can be represented as above, this defines the projection of $L_k(Z^n)$ onto $L_{k-1}(Z^{n-1})$. If the obstruction for the surgery problem $(N, f \mid N, \nu_0, F \mid N)$ vanishes, we may assume $f \mid N : N \longrightarrow X$ is a homotopy equivalence. So if the surgery problem $(M, f, \nu_0 \oplus \nu_1, F)$ is treated relative to N, we get an element in $L_k(Z^{n-1})$. Conversely, every element of $L_k(Z^{n-1})$ may be so represented. This describes the injection of $L_k(Z^{n-1})$ into $L_k(Z^n)$. For any surgery problem (N, g, ν_0, F_0), by a product with S^1 we have a new problem $(N \times S^1, g \times id, \nu_0 \oplus \nu_1, F_0 \oplus F_1)$ where F_1 is the trivial framing of $TS^1 \oplus \nu_1$. This defines an element in $L_k(Z^n)$ and yields the splitting homomorphism.

For the nonorientable case see [Wa].

(2): By [Su],

$$NM_{TOP}(X, N) = [X, N; G/TOP, *],$$

The 5-skeleton of G/TOP is $K(Z,4) \times K(Z_2, 2)$, hence it follows.

(3): We need to compute $S_{TOP}(S^1 \times D^j; \partial)$ or $S_{TOP}(S^1 \tilde{\times} D^j; \partial)$. It can be shown in this case that $NM_{TOP} \longrightarrow L_4$ is an isomorphism. Therefore, S_{TOP} contains only one point. For a general theorem, see [FQ]. $\square$

Chapter 3

Codimension two disk theorem

In this chapter, the main technical theorems that will be used to classify 4-manifolds are proved. Both disk theorems are extracted from Chapter 10 of [FQ]. For the 5-dimensional case, a correction was neccesary [St1].

3.1 The Whitney move

3.1.1 Framing

Let $\alpha : S^1 \longrightarrow M^3$ be an embedding, and $\nu_{S^1 \to M^3}$ be the 2-dimensional normal bundle of S^1 in M^3. Assume that it is trivial. Then a framing of α is a homotopy class of trivialization of ν. Fixing a framing of α, then any other framing differs by a homotopy class of maps $d : S^1 \longrightarrow SO(2)$. Therefore, with respect to a fixed framing, any other framing is determined by an integer.

Let $\alpha : (D^2, \partial D^2) \longrightarrow (M^4, \partial M)$ be a proper immersion($\alpha(intD^2) \subset$ *intM and* $\alpha |_{\partial D^2}: \partial D^2 \longrightarrow \partial W$ is an embedding), then α determines naturally a framing of $\alpha : \partial D^2 \longrightarrow \partial M$ as follows: $\nu_{D^2 \to M^4}$ is trivial because it is a bundle over D^2. Note that any trivialization $\tau : \nu \cong D^2 \times R^2$ induces uniquely a trivialization $\nu |_{\partial D^2} \cong \partial D^2 \times R^2$ on the boundary. In fact, the difference of any two induced framings correspondes to a map $d : \partial D^2 \longrightarrow SO(2)$. Since d can be extended to a map on D^2, the difference map d is homotopic to 0. Hence the two framings coincide. Then there is a natural framing of $\alpha\ |_{\partial D^2}$ in ∂M^4(called *the framing determined by* α). The framing determined by α is characterized by the following property: any two maps α', α'' homotopic to α which differ on ∂D^2 by pushing off along this framing have integral intersection number 0.

The framing is closely related to the intersection and self intersection numbers. First we have the following formula: let x be an immersed 2 sphere in a 4-manifold whose normal bundle has Euler number χ, then in Λ

$$\lambda(x, x') = \mu(x) + \mu(x') + \chi.$$

where x' is a section of the normal bundle. Let x be an immersed disk with a fixed framing on the boundary. Let χ be the rotation number of this framing with respect to the normal bundle of the immersion restricted to the boundary(note this is not the framing determined by the map), then in Λ

$$\lambda(x, x') = \mu(x) + \mu(x') + \chi,$$

where x' is a parallel to the fixed framing on the boundary. Taking the coefficients at the identity, we have

$$\lambda_1 = 2\mu_1 + \chi.$$

By this formula, take the framing determined by the immersion as the fixed framing of x, by definition $\lambda_1(x, x') = 0$. Therefore, the rotation number with respect to the normal bundle of the immersion is $-2\mu_1(x)$ as integers.

Example 1 *The immersion of a disk with a single double point determines a framing of -2.*

Example 2 *A k-framing attaching of 2-handles on a knot in S^3 determines its framing of the core disk as k with respect to the normal bundle.*

Example 3 *Consider the Chern manifold which is built as follows: take a right handed trefoil in S^3, attach a 2-handle with framing +1, then the boundary is a homology 3-sphere. Cap off the boundary homology 3-sphere by a contractible 4-manifold, and obtain the Chern manifold. Then there is an immersed disk with one double point which has an embedded Whitney disk with wrong framing.*

In the next section, we will see there is always a framing problem for the Whiney move, an embedding Whitney disk alone is not enough.

3.1.2 The Whitney move

The Whitney trick is a tool to introduce algebra into topology. In dimension 5 or higher, it works very well by general position. But in dimesion 4, general position is not enough and it is very delicate.

A convenient model for this move is as follows: in the 2-disk D^2, consider two proper embedded arcs α and β with only two intersection points, together with a spanning disk W in $intD^2$ with boundary on the arcs. In the 4-ball $D^4 = D^2 \times I \times I$, where I=[-1, +1], let $A = \alpha \times I \times 0$, $B = \beta \times 0 \times I$, then A and B are unknotted 2-disks in D^4 which intersects transversly in two points of opposite sign. They can be isotoped to be disjoint, moving only points close to $W = W \times 0 \times 0$ in $intD^4$, by the Whitney move which uses W as a guideway.

Suppose in an oriented 4-manifold M one has surfaces A and B which are connected and embedded (for example, one has an embedded portion of immersed surfaces having two transverse intersection points p and q of opposite sign). Let α and β be paths in A and B, respectively, joining p and q, and suppose W is an immersed disk in M whose boundary is attached to $\alpha \cup \beta$. Assuming everything is generically positioned, then W may have self-intersection in its interior. We wish to use W as a Whitney disk to remove p and q. But now we have some problems. They are (i) the framing of W may be wrong. (ii) intW may have self-intersections. (iii) intW may intersect $A \cup B$ (also intW may have unwanted intersection with some other surface C). To resolve (ii), push W off itself through A or B at the expense of creating two additional intersection points of intA (or B) with intW for each selfintersection of W. To resolve (iii), we again push A off itself through A or B at the expense of selfintersection in A or between A and B. To see the problem of (i) needs some work. The Whitney disk determines a framing (called the Whitney framing) which can be described as follows: since W is immersed, there is an immersion $\pi : D^4 \longrightarrow M$ of our model into a neighbourhood of W, carrying $\hat{W}$ onto W (we use $\hat{}$ over the model disks). Then W has the correct framing (as a Whitney disk) if in addition we can make π carrying $\hat{A}$ into A and $\hat{B}$ into B. Either one or the other is easily arranged, but there is a potential obstruction to achieve both simultaneously. For example if we look at the circles $\pi^{-1}(A) \cap \partial D^4$ and $\pi^{-1}(B) \cap \partial D^4$, their union is a link, but may look as a twisted Hopf link in ∂D^4 even though both of these circles are unknotted and their algebraic intersection number is 0. To remedy this framing problem, twist W at an arbitrary point of $\partial W - \{p, q\}$. Each twist has the effect of introducing a new intersection point between intW and A or B. Finally notice that to resolve (ii) (iii), we use regular homotopy, so it does not destroy the correct framing if we have.

Another nice description of the Whitney move is as follows: if the Whitney framing is correct, then we get an unlink in ∂D^4 which is a slice link. It bounds two disjoint slice disks in D^4. Remove everything of A and B inside D^4, replace them by glueing in the slice disks. Since D^4 is contractible, we may recover all the homotopy data.

Suppose there are additional disks $C_1, C_2, \cdots, C_k$ present of the form $C_i = p_i \times I \times I$, where $p_i \in intW$ in the standard model. Then the boundary circles of A,B, C comprise a link which is the Borromean ring for k=1. Initially the C_i's are disjoint from A and B, but after the Whitney move there will be intersections. Using finger moves, C can be kept free of either A or B, but not both.

There is an analogous move for accessory disks (disks passing through one double point). If the regular neighbourhood of the disk intersects the boundary of 4-ball is a slice knot, then we say the framing is correct. Otherwise it is wrong. As observed by Freedman,besides the unknot, the Stevedore's knot has the coorect framing, too.

3.2 The disk theorem for dimension 4

In[F1], M. Freedman proved the best possible disk theorem so far. There are two restrictions in the theorem. First the fundamental group must be good; secondly, there is the "even dual" condition. The following generalized disk theorem of [FQ] has dealt with this condition. More or less, it is reflected in the Kirby-Siebenmann invariant of the target manifold.

In the following, in order to distinguish from the transverse sphere, we will use the terminology *dual 2-sphere* . They are the same as transverse spheres except they might not be framed. Recall in the definition of transverse spheres, they are framed.

Theorem 3 *Let* $h : \{A_i\} = A \longrightarrow M^4$ *be a proper immersion of 2-disks with algebraic intersection and self intersection numbers* $0 \in Z[\pi_1]$. *Moreover, assume* *there is a family of dual 2-spheres $\{\alpha_i\}$ of A. If $\pi_1 M$ is good and has no 2-torsion, then there is an obstruction $km(h) \in Z_2$ which vanishes iff h is regularly homotopic rel∂ to an imbedding. Moreover, if km(h)=1, then h is regularly homotopic rel∂ to an embedding in $*M$.*

Finally, if all the dual 2-spheres can be framed, then km(h)=0.

Remark1: for the *-operation, see chapter 4.3.

Remark2: if all the dual 2-spheres can be framed, then this is theorem 5.1B of [FQ]. If km(h)=1, then $\omega_2 : \pi_2(M) \longrightarrow Z_2$ can not be 0. Since $\pi_1(M)$ is good, hence *M always exists.

To motivate our definition of km, we first consider the case of a single disk. Suppose A is a single disk with selfintersection number 0 and a dual 2-sphere, then choose immersed Whitney disks $\{B_i\}$ of all pairs of self intersection points for a fixed pairing of the selfintersection points. Define $km(B_i) = 0$ if the Whitney framing of B_i is correct, otherwise it is 1. Then define $km(A) = \sum km(B_i) \; mod\, 2$. As an exercise, we can show that if km(A)=0, then A is regularly homotopic to an embedding rel boundary.

Definition 4 *Choose a family of dual 2-spheres for A_i, denoted by $\alpha = \{\alpha_i\}$, choose a complete set of Whitney disks for all intersections and self intersections among A, denoted by $W = \{W_{ij}\}$. Define*

$$km(W_{ij}) = \omega_2(\alpha_i)\omega_2(\alpha_j) \sum_k \mid W_{ij} \cap h(A_k) \mid \omega_2(\alpha_k),$$

where W_{ij} is a Whitney disk with boundary arcs on A_i, A_j, k is the indices of A_*. Then let $km(h) = \sum_{ij} km(W_{ij}) \; mod\, 2$.

km(h) is well-defined [FQ]. But if there is 2-torsion in $\pi_1 M$, then there is one more case to check [St3]. It is clear that if h is homotopic rel∂ to an embedding then km(h)=0. Therefore, the first part of the theorem is equivalent to if km(h)=0 then h is homotopic to an embedding.

Proof: If km(h)=0, by piping one Whitney arc to the other, we may assume that km(h)=0 for each Whitney disk. By summing with the dual 2-sphere, $\{W_{ij}\}$ is disjoint from $A(D_*)$. If a disk W_{ij} is changed by an even number of sums with spheres with $\omega_2 \neq 0$, then the normal bundle of the results differs on the boundary by an even number of twists from the Whitney framing. Interior twists in 1.3 of [FQ] can be used to correct this. If there are an odd number of such spheres then by $km(W_{ij}) = 0$ and the definition of $km(W_{ij})$ one of the boundary arcs must be on a disk with framed transverse sphere, say $A(D_i)$. Since $A(D_i)$ has framed transverse sphere this intersection can be removed without disturbing the framing . Therefore, there are Whitney disks with interior disjoint from $A(D_*)$.

Now we can prove the theorem by theorem 5.1A of [FQ]. For each such Whitney disk, consider the linking torus of an intersection point as a transverse capped surface. Use $\{\alpha_i\}$ to get caps for these disjoint from A, and contract to get algebraic transverse sphere for the Whitney disks, with algebraically trivial intersection and selfintersection. Since $\{\alpha_i\}$ are added to caps in a capped surface which is then contracted, so each α_i enters algebraically 0 times. This implies twists in the normal bundle cancell out.

Denote by M_0 the complement of an open regular neighbourhood of A in M. Then the Whitney disks and the algebraically transverse spheres give an immersion in M_0 satisfying the hypothesis of theorem 5.1A. The conclusion of theorem 5.1A gives the required Whitney disks. Use these for Whiney moves to produce a regular homotopy of A to an embedding.

Now consider the case km(h)=1. Let $f : CP^2 \longrightarrow *CP^2$ be the canonical homotopy equivalence, then $f_0 : CP_0^2 \longrightarrow *CP^2$ restricted to the core disks can not be homotopic to an embedding. Therefore, $km(f_0) = 1$. Consider $h \cup f_0$: $\{A_i\} \cup core\ disk \longrightarrow M\sharp(*CP^2)$. If km(h)=1, then $km(h \cup f_0) = 0$. This gives an embedding of A in the complement of $CP_0^2 \subset M\sharp(*CP^2)$, which is $(*M)_0$ by definition. Therefore, h is homotopic rel boundary to an embedding in $*M$. □

3.3 The disk theorem for dimention 5

3.3.1 Intersections of 3-disks in 5-manifold

Let $A : \cup D_j^3 \longrightarrow W^5$ be a proper map of a disjoint union of 3-disks into a 5-manifold W in general position, and transverse to each other. Then the intersections are circles and arcs of double points. They are disjoint from boundaries of the disks since A is an embedding there. A is immersed except at isolated cusps occuring at the ends of arcs of double points. The preimage of the double locus in $\cup D_i$ is an oriented link. The orientation indicates how two circles or arcs of one circle are identified in W^5. Since the orietation plays no role in the following, we will suppress it. For the double circle, the preimage is a 2-fold covering, so either two circles trivially cover it (called Type II circle) or a single circle going twice (called type I circle). For the double arc, a single circle 2-fold branch-covered the arc branched over the 2 ends. Therefore, the intersections are classified into 3 types according to the preimage.

For each double circle or arc or a double point on the double locus, there is an associate group element in $\pi_1 W$ defined as follows: fixed a base point in each 3-disk D_i and choose 2 points in the preimage of a point on the double locus, say these two points are in D_j and D_k, join the base point of D_j to one point in the preimage , jumping to the other one in D_k and going back by a path connecting the the base point of D_k. Then the image of this arc under A is a closed loop in W^5 which defines an element in $\pi_1 W$. This element does not depends on the chosen point of the double locus, it only depends on the double circle or arc. If we follow the same path from D_k to D_j instead of from D_j to D_k, then the element changes to its inverse. Hence it is well defined up to conjugate and inverse. For the double arc, the group element is the identity. For Type I circle, since the selfintersection circle itself is a representative and the double cover is nullhomotopic, it follows that the group element is always a 2-torsion. For Type II circles, we can associate the element to the preimage circle by remembering which one is the first to define the group element. Then if one component has group element g, then the other component has g^{-1}.

As usual, we want to remove these intersections as much as possible. For the double arc, it can always be changed into Type I or II by pulling the end together,see [FQ] for details. Another operation is a version of the Whitney move. It will be used to change one type of circle to the other. We will call it reconnection. This operation is introduced in [FQ], but we have to be more precise. Suppose x and y are points of intersection between $A(D_i)$ and $A(D_j)$ which has dual 2-spheres such that (1) the associate group element of x, y agrees to get a Whitney disk. (2) choose band to tell which direction to thicken and the two bands glued together give $S^1 \times R$ instead of the Mobius band. (3) signs of intersection is compatible if they are on one component. (4) the framing modulo 2 is correct. Then the double locus can be reconnected at the points x, y by a 4-dim Whitney move cross R. If the framing is not correct, then it can be fixed by introducing twists in the bands. Note one full twist in one band is not exactly the same as half twists in both bands.

Another fact we need is that by introducing double arcs, we can get any number of double circles with identity associate group element.

3.3.2 The Freedman-Quinn's move

This is an ambient surgery to remove a double circle by cobordism. Let $h : A \longrightarrow Y$ as above and have only type II intersections. Suppose a circle of intersection has preimage two circles $r \subset D_i$ and $r' \subset D_j$, and r is unkotted and unlink. Let $C \subset D_i$ be an embeded disk with boundary r and interior disjoint from double points of A. This data will be used to construct a 4-dim handlebody with D_i' at one end, and a map into Y extending A. In some cases, the boundary is a 3-disk and the double circle is removed by the surgery.

Starting with $D_j \times I$, and add a 2-handle on $r' \times \{1\} \subset D_j \times \{1\}$. Map this to the normal disk bundle of the image of D_i, restricted to the 2-disk C. This singles out a specific framing of $r' \times \{1\}$ on which to attach the handle. To undo the first handle attachment, begin with a cocore of the first handle (a fibre of the disk bundle over C). By perturbing and summing with a dual 2-sphere α_i, it becomes an embedding. By extending the normal bundle of the boundary circle in the boundary of the handlebody, we get a 2-handle. This extension is not unique. This handlebody gives a cobordism from g on D_j to a map on a 3-manifold which has the same selfintersections and intersections with other D_* as D_j, except $r \cup r'$ has been eliminated. Since there are only 2-handles attached, the $Z[\pi_1]$ class has been preserved, so we can simplify the situation provided the new 3-manifold is a 3-disk. When all the dual 2-spheres can be framed, this can be achieved. See [FQ].

Before we discuss another situation which the new 3-manifold is also a 3-disk, let us define the rotation number. If $r \subset D_i$ is the preimage of $g(D_j)$, then the normal bundle of r in D_i is the restriction of the normal bundle of $g(D_j)$ in

Y. The contractibility of D_j defines a trivilization of this bundle. But r is also the boundary of an immersed disk in D_i, which also gives a trivilization of the bundle. These trivilizations differ by a rotation in $\pi_1 O(2) = Z$ and this is defined to be the rotation number of r. Notice given a pair of dual circles r, r', the rotation numbers of them are not the same. It is also true that we can arrange that the rotation number to be $\omega_2(\alpha_i) mod 2$. If $\omega_2(\alpha) = 1$, then the rotation number can not be easily arranged to be 0. But actually, this can be changed to 0 [FQ]. Hence without loss of generality, we way assume the rotation number is 0.

Now suppose $r' \subset D_j$ bounds an embeded disk (not necessarily disjoint from the other double points). This disk defines a trivilization of the normal bundle, which differs by the rotation number from the framing used to attach the first 2-handle. If these framings agree, then the new 3-manifold is a 3-disk. But after the cobordism, it introduces a twist into the double circles passing through r. This is what missed in chapter 10 of [FQ].

3.3.3 The disk theorem in dimesion 5

In this section, we consider the disk theorem in dimension 5. Given a 5-dimensional manifold Y with $\pi_1 = \pi_1 Y$ and a homology class $\beta \in H_3(Y, \partial Y; Z[\pi_1])$, we are interested in representing β by a π_1-negligible embedded 3-disk. The same argument works for 3-sphere. By the relative Hurewicz's theorem, β can always be represented by a map (D^3, ∂) to Y. In the following, we assume that the map is always an embedding on a neighbourhood of ∂D^3.

Theorem 4 *Let $h : D = \{D_i^3\} \longrightarrow Y$ be a proper map of a union of 3-disks in general position and transverse to each other whose intersection and selfintersection numbers are $0 \in Z[\pi_1]$, and with dual 2-spheres α_i, and if $\pi_1 Y^5$ has no 2-torsion,* *then there is an obstruction km(h)* $\in H_1(Y; Z_2)$ *such that if km(h)=0 there is a* π_1*-negligible embedding* $h' : D \longrightarrow Y$ *such that* $h'(D)$ *and h(D) represents the same homologous class in* $H_3(Y, \partial; Z[\pi_1 Y])$.

In particular, if all the dual 2-spheres of D are framed, then km(h)=0.

The obstruction will be defined in the proof and it is due to R. Stong[St1]. In most cases, we have $\omega_2 = 0$ on $\pi_2 Y$ or ω_2 does not vanish on the perpendicular subspace of $\pi_2 Y$, then km(h)=0.

Proof: By the discussion above, we way assume there are no double arcs. Since there are no 2-torsion in $\pi_1 Y$ except the identity, then all type I circles have associate group element identity. By introducing this kind of intersection if necessary, we may assume there are even number of them. By using the reconnection, they can all be converted into Type II circles. Therefore, we may assume there are only Type II circles. Now we will use Freedman-Quinn's move to eliminate all double circles if km(h)=0. Notice up to now, all the moves are regular homotopy. In this case, each circle in the preimage is paired with another one, and they will be called dual to each other. Therefore, the preimage of the intersection circles is an orinted link in D_i. There are immersed disks bounding the circles, whose intersections are all clasps. The clasp can be pulled apart as [FQ]. After undo all the clasps, we get some unknotted big circles with many small circles linked to each big one.

Lemma 1 *Let A and A′ be two dual big circles, then the number of small linking circles to A plus the number of linking circles to A ' is even.*

All this circles can be simplified to a single Hopf link with group element g. Then the image of g in $H_1(W, Z_2)$ is the obstruction km(h). Actually, any square or commutator in $\pi_1 Y$ can be the associate group element for a Hopf link by introducing trivial intersections. km(h) is well defined [St1]. If km(h)=0, by introducing a Hopf link with identity group element, this pair can be removed. □

Remark: It seems more than a coincidence that km is in the same group as the ks invariants for both dimensions 4 and 5.

Chapter 4

Classification

In this chapter, the complete classification for 4-manifolds with $\pi_1 = 0, Z$ is obtained. First, the stable classification is obtained using the work of M. Kreck and P. Teichner. Then the classification theorem is proved.

4.1 Stable Classification

Before the breakthrough of M. Freedman, there is a stabilized version of 4-manifold theory [CS]. This theory is much easier. A stabilization of M is the manifold $M\sharp k(S^2 \times S^2)$, where k is an integer.

Definition 5 *Let M, N be two locally oriented 4-manifolds, M is called stably homeomorphic to N if there exists natural number r, s such that $M\sharp r(S^2 \times S^2)$ is homoemprphic to $N\sharp s(S^2 \times S^2)$.*

Here the connected sum has to be formed compatibly with the local orientation.

The following method of M. Kreck to determine the stable homeomorphic type of closed 4-manifolds is outlined and the stable classification of 4-manifolds with fundamental group Z is computed. For details of the theory see [Kr], [Te1]. The basic notion of the theory is ξ-structure and normal 1-type.

Definition 6 *Given a fibration $\xi : X \longrightarrow BO$, a ξ-structure of a 4-manifold M is a lift of the stable normal Gaussian map $M \longrightarrow BO$ over ξ, up to fiber homotopy.*

The normal 1-type of a 4-manifold M is the fiber homotopy type of $\xi : X \longrightarrow BO$ such that

(1): there exists a normal structure $\tilde{\nu} : M \longrightarrow X$ which is a 2-equivalence.

(2): ξ is 2-coconnected, i.e, $\xi : \pi_i X \longrightarrow \pi_i BO$ is a monomorphism for i=2, and an isomorphism for $i > 2$.

By obstruction theory, the normal 1-type exists uniquely. Now the stable classification goes as follows:

Step I: determing the normal 1-type of M. The normal 1-type of a 4-manifold is completely determined by $\pi_1 M, \omega_1 M, \omega_2 M$ and $\omega_2 \tilde{M}$.

Step II: translating into a bordism problem.

Step II: computing $\Omega_4(\xi)$ and the linear action of $Aut(\xi)$.

The theory for smooth and topological stable classification differs by the Kirby-Siebenmann invariant.

Now given π_1, ω_1, we want to classify all 4-manifolds with these data up to stable homeomorphism. First the right ξ -structure is called 1-universal fibration. A 1-universal fibration is a fibration $\xi : X \longrightarrow BO$ such that it is 2-coconected. The 1-universal fibrations are in 1-1 correspondence to $\omega_2 \in H^2(\pi_1, Z_2) \cup \infty$, where ∞ denotes the case $\omega_2 \tilde{M} \neq 0$.

For the stable classification, we have the following theorems:

Theorem 5 *(1): Two closed nonspin orientable 4-manifolds with $\pi_1 = Z$ are stably homeomorphic iff their signature and Kirby-Siebenmann invariants ks are the same.*

(2): The stable homeomorphism type of a spin orientable closed 4-manifolds with $\pi_1 = Z$ are determined by the siganture only.

Remark: for the definition of *-operation, see section 4.3.

The first part of the theorem is in [Kr] and the second part is an easy exercise for this theory. Recall that in the spin case ks is determined by the signature.

Theorem 6 *Let M be a closed connected nonorientable 4-manifold with $\pi_1 = Z$, then*

(1): If $\tilde{M}$ is not spin, then M is stably homoemorphic to one of the following 4 manifolds: $S^1\tilde{\times}S^3\sharp CP^2$, $S^1\tilde{\times}S^3\sharp * CP^2$, $S^1\tilde{\times}S^3\sharp CP^2\sharp CP^2$ or $S^1\tilde{\times}S^3\sharp CP^2\sharp * CP^2$. They are determined by ks and mod 2 euler number.

(2): If $\tilde{M}$ is spin, then M is stably homeomorphic to $S^1\tilde{\times}S^3$ or $S^1\tilde{\times}S^3\sharp E_8$.

(3): If $\tilde{M}$ is spin, then the Kirby-Siebenmann of M is a homotopic invariant and it is the same as $[\lambda] \in L_4(Z,-)$, where $[\lambda]$ is the value of the intersection form on $\pi_2(M)$ as an element in $L_4(Z,-)$.

Note that $\pi_2(M)$ is free and μ is determined by λ.

Proof: (1): In this case, the 1-universal fibration is given by $\xi = \rho \oplus \eta : BSO \oplus B\pi_1 \longrightarrow BO$, where η is a line bundle with $\omega_1(\eta) = \omega_1(M)$. There is a spectral sequence with E_2-term $H_p(B\pi_1, \Omega_q^{\tilde{SO}})$ converging to $\Omega_{p+q}(\xi)$, where the coefficients are twisted by $\omega_1 : \pi_1 \longrightarrow Z_2$ as follows: for any abelian group A, include Z_2 into $Aut(A)$ by the map $a \rightarrow -a$. Then it is fairly straightforward to compute.

(2): In this case, the 1-universal fibration is given by $\xi = \rho \oplus \eta : BSpin \oplus B\pi_1 \longrightarrow BO$, where η is a line bundle with $\omega_1(\eta) = \omega_1(M)$ and $\omega_2(\eta) = 0$. There is a spectral sequence with E_2-term $H_p(B\pi_1, \Omega_q^{TO\tilde{P}Spin})$ converging to $\Omega_{p+q}(\xi)$. Then it also a spectral sequence computation.

(3): Notice that both the Kirby-Siebenmann invariant and the value of an intersection form in $L_4(Z,-)$ is a stable invariant. So it only necesasary to check on the generators. Then by (2) and the fact that $L_4(Z,-)$ is generated by the intersection form of $S^1\tilde{\times}S^3\sharp E_8$, they agree on generators. □

4.2 Elementary properties

In this section, we collect some elementary properties for 4-manifolds with free fundamental group. Denote the free group with k generators by F_k for $k = 0, 1$. For k=0, it is the trivial group, and k=1 it is J. These properties will be used later to prove the classification theorem.

Proposition 4 *Let M^4 be a closed connected 4-manifolds with $\pi_1 = J$, then*

(1) $\pi_i(M) \longrightarrow H_i(M,Z)$ is surjective for i=0, 1, 2, 3. In particular, all classes of $H_2(M,Z), H_3(M,Z)$ are spherical.

(2) $\pi_2(M)$ is a free $Z[J]$-module.

(3) The intersection form on $\pi_2(M)$ is nonsingular.

(4) M^4 is weakly even or weakly odd iff M^4 is even or odd.

Proof: (1) for i=0, 1,2, it is trivial by Hurewicz's theorem.

For i=3, using Serre's spectral sequence for $\tilde{M} \longrightarrow M \longrightarrow S^1$, thus $H_3(\tilde{M};Z) \longrightarrow H_3(M;Z)$ is onto. On the other hand, by Hurewicz's theorem for $\tilde{M}$, $\pi_3\tilde{M} = \pi_3 M \longrightarrow H_3(\tilde{M};Z)$ is onto.

(3) As a corollary of Seshadr's theorem [Ba], a $Z[F_k]$-module is free if and only if it is projective. Since stably freeness implies projectiveness, it is sufficient to prove stably freeness. For any 4-manifold, by connecting sum with CP^2 or $*CP^2$, we may assume it is smooth, odd up to stabilization by connecting sum with $S^2\times S^2$. It follows that up to stabilization with CP^2, $*CP^2$ any 4-manifold with $\pi_1 = J$ is stably diffeomorphic to $S^1 \times S^3$ or $S^1\tilde{\times}S^3$ which obviously have free π_2.

(3) We have the following exact sequence which comes from the spectral sequence of the universal covering with $\Lambda = Z[\pi_1]$ coefficients

$$0 \longrightarrow H^2(\pi_1, \Lambda) \longrightarrow H^2(M, \Lambda) \xrightarrow{b} Hom_\Lambda(\pi_2 M, \Lambda) \longrightarrow H^3(\pi_1, \Lambda) \longrightarrow 0$$

For π_1 free, $H^2(\pi_1, \Lambda) = 0$, $H^3(\pi_1, \Lambda) = 0$, hence

$$b : H^2(M, \Lambda) \longrightarrow Hom_\Lambda(\pi_2 M, \Lambda)$$

is an isomorphism. Since the intersection form λ is the composition of the inverse of Poincare duality with Λ-coefficient

$$\pi_2(M) = H_2(M; \Lambda) \rightarrow H^2(M; \Lambda)$$

and b, hence it is nonsingular.

(4) One direction is always true. The other follows from definition.

□

4.3 *-operation

By Rochlin's theorem, for closed topological spin 4-manifolds, the Kirby-Siebenmann invariant is a homotopy invariant. But there are homotopic 4-manifolds with different Kirby-Siebenmann invariant. The famous pair is the CP^2 and the Chern manifold $*CP^2$. In chapter 10 of [FQ], an operaton is defined to change the Kirby-Siebenmann invariant while preserving the homotopy type.

Definition 7 *Suppose W is a 4-manifold, if $\omega_2 : \pi_2 W \longrightarrow Z_2$ is trivial, then define *W=W. If $\omega_2 : \pi_2 W \longrightarrow Z_2$ is nontrivial, the *W is determined by the following construction: $(*W)\sharp CP^2$ is homeomorphic to $W \sharp * CP^2$ by a homeomorphism which* *preserves the decomposition of π_2, where $*CP^2$ is the Chern manifold which is homotopic to CP^2 but has ks=1.*

Note that if *W is different from W, then it has opposite Kirby-Siebenmann invariant because Kirby-Siebenmann invariant is additive under connected sum. Also there is a canonical homotopy equivalence from $*W_0$ to W_0 which is an isomorphism on the boundary.

If $\pi_1 W$ is good, then the *-operation exists [FQ] and is well-defined [St3].

Rem: For the case $\omega_2 : \pi_2 W \longrightarrow Z_2$ trivial, there are also manifolds which has the same homotopy type but different ks. Unfortunately, this is not unique. The following example is due to P. Teichner [Te2]: let E be the one point compactification of the tangent bundle of RP^2, then there is a E' which is homotopic equivalent to E but has different ks. The manifolds $E\sharp E$ and $E'\sharp E'$ are both * manifolds (homotopic but has different ks) for $E\sharp E'$, and they are not homoemorphic. Actually they are even not stably homeomorphic.

4.4 Classification for $\pi_1 = 0$

If M is a compact oriented 4-manifold, then the intersection numbers define a symmetric bilinear form $\lambda : H_2(M;Z) \times H_2(M;Z) \longrightarrow Z$. If $\partial M = \emptyset$, then by Poincare duality, the adjoint $H_2(M;Z) \longrightarrow H_2(M;Z)^* = Hom_Z(H_2(M;Z))$ is an isomorphism. Therefore, it is nonsingular. For the classification another additional piece of infomation is given by the Kirby-Sienbenmann invariant, which is in genaral not a homotopy invariant. The first complete classification is the following result by Freedman [F1]. Here we offer a new proof.

Theorem 7 *(1).***Existence:** *Suppose (H, λ) is a nonsingular symmetric bilinear form on a finitly generated free Z-module, $k \in Z_2$, and if λ is even, then assume*

$k \cong \frac{sign\lambda}{8}$, then there is a closed oriented 1-connected manifold with form λ and Kirby-Siebenmann invariant k.

*(2).***Uniqueness:** *Suppose M, N are closed and 1-connected 4-manifolds, $h : H_2(M;Z) \longrightarrow H_2(N;Z)$ is an isomorphism which preserves the intersection form and ksM=ksN, then there is a homeomorphism $f : M \longrightarrow N$ such that $f_* = h$.*

Proof: First, we prove the existence. Let B^4 be a 0-handle. Find a framed link L such that the linking matrix of L in $S^3 = \partial B^4$ represents the form; adding 2-handles to B^4 according to the framing of L, then the resulted 4-manifold M_L^4 is a simply connected, smooth 4-manifold with boundary. Since $H_1(M_L, Z) = 0$, by the exact sequence of the pair (M_L, ∂), we have

$$0 \longrightarrow H_2(\partial;Z) \longrightarrow H_2(M_L;Z) \longrightarrow H_2(M_L,\partial;Z) \longrightarrow H_1(\partial;Z) \longrightarrow 0.$$

Since the form is nonsingular, by Poincare duality the map $H_2(M_L;Z) \longrightarrow H_2(M_L,\partial;Z)$ is an isomorphism. Hence the boundary 3-manifold is a homology 3-sphere. Every homology 3-sphere bounds a contratible 4-manifold [FQ]. Therefore, add such a contractible 4-manifold with the homology 3-sphere as boundary, we have a simply connected closed topological 4-manifolds. It is easy to check that the 4-manifold has the right intersection form λ.

By the above construction, for each form one model 4-manifold is constructed. If the form is odd, another 4-maniold with opposite Kirby-Siebennmann invariant can be obtained by the *-operation.

Now we prove the uniqueness. Without loss of generality, assume M is the model we built above: $M = B^4 \cup_L$ 2-handles$\cup C = M_L \cup C$, where C is the contractible 4-manifold. In M, there is a canonical basis for $H_2(M;Z)$. It is represented by the cores of the 2-handles with the boundary circle capped off by a singular disk in B^4. The manifold M_L is homotopically a wedges of 2-spheres. Choose the basis of $H_2(N;Z)$ as the image of the canonical basis of $H_2(M_L)$ under h, then h is the identity matrix and the intersection form of N is the same as that of M. Define a continuous map from M_L to N by sending the canonical basis to the corresponding basis of $H_2(N;Z)$. Now we want to make this into an π_1-negligible imbedding.

By a homotopy and general position, we way assume that $h : M_L \longrightarrow N$ is an embedding on the 0-handle. Delete the 0-handle and denote the complement by M_0, similiarly delete the image of 0-handle in N denoted by N_0. Denote the core disks of the 2-handles by $\{D_i\} = D$. The boundaries of D are on the boundary of the 0-handle with a fixed framing determined by the attaching of 2-handles. Approximate the map on D by an immersion f using lemma 1.3 [FQ]. Then there might be some rotations rel boundary with respect to the fixed framing of boundaries. Let χ_i be the rotation numberof the disk D_i with respect to the fixed framing, let A_i be the immersed 2-sphere $D_i \cup f(D_i)$ in $M_0 \cup_\partial N_0$, then we have

$$2\mu + \chi_i = \lambda(A_1, A_1).$$

Since f preserves intersection numbers, hence $\lambda(A_i, A_i)$ is even. It follows that the rotation number χ_1 is even. Therefore remedy this by interior twist if necessary, the framing is correct. By the nonsingularity of the intersection from on $H_2(M;Z)$, there are dual 2-spheres for D. By the disk thereom 3.2 , there is an obstruction $km(f) \in Z_2$ to homotope f to a π_1-negligible embedding.

If km(f)=0 (e.g. if the form is even) , then f is homotopic to a π_1-negligible embedding f' rel boundary. By thickening the core disks, it follows that there is an embedding of M_0 in N_0. Filling in the 0-handle, we get an embedding of M_L in N. By π_1-neglibility, and isomorphism on H_2, the complement of M_L in M is simply connected, no homology, therefore it is a contractible 4-manifold with the same boundary as C. But such 4-manifolds are unique[FQ], so the embedding can be extended to an embedding of M. By construction it induces h on H_2.

If km(f)=1, then there is an embedding in $*N \neq N$, but this implies that M is homeomorphic to *N. Again this implies ksM≠ksN which contradicts the assumption. Therefore, ks(f)=0 if ksM=ksN which ends the proof as above. □

Rem: note that the final embedding produced might not be homotopic to h. The embedding in the last step is extended to the contractible piece by uniqueness of such manifold, this is not a homotopy in general.

4.5 Classification for orientable $\pi_1 = Z$

Suppose M is a closed oriented 4-manifold with $\pi_1 = Z$. To distinguish from the coefficient Z in the group ring, we will use J to denote Z. It has been shown that $\pi_2(M)$ is a free $Z[J]$-module. When M is orientable, the intersection form is a nonsingular hermitian form on π_2. The analogue of section 4.4 is the following

Theorem 8 *(1).* **Existence:** *Suppose (H, λ) is a nonsingular hermitian form on a finitely generated free Z[J]-module, $k \in Z_2$ and if λ is even, assume $k \cong \frac{sign\lambda}{8}$, then there is an oriented closed manifold with $\pi_1 = Z$, intersection form λ and Kirby-Siebenmann invariant k.*

(2). **Uniqueness:** *Suppose M, N are closed oriented 4-manifolds with $\pi_1 = Z$, $h : \pi_2 M \longrightarrow \pi_2 N$ is a Z[J] isomorphism which preserves the intersection form and ks(M)=ks(N), then there is a homeomorphism inducing the identification of π_1, preserves the orientation and $f_* = h$.*

Remark: In our situation, λ is weakly even or odd is the same as the even or odd for the integral intersection form S. So by $sign\lambda$ we really means the signature of S.

Before the proof, we discuss the link theory in the solid torus. To build a standard model for 4-manifolds with $\pi_1 = Z$, we will attach 2-handles on a link in a solid torus. Let's define the linking matrix for links in the solid torus.

Denote the standard solid torus by $T = S^1 \times D^2$. Our link in T always satisfies the following conditions: L is oriented, each component of L is homotopically trivial, equivalently, each component as a map from S^1 to T has degree 0. Given such a link in T, the linking matrix is defined as follows: fix a base point of T and a base point on each component of the link, and a path from the base point of the solid torus to the base point on each component of the link. The linking number of two components as an elements in $Z[J]$ is defined in the following way. Bound one component with an immersed disk and put it in general position with the other component, list all intersection points of the bounding disk with the other component. For each intersection point, follow the path from the base point of the solid torus to the base point of the bounding component, then join the base point of this bounding component with the intersection point with an arc on the disk. Follow the other component along the positive direction to the base point of this component. Finally, follow the path from the nonboundng component to the base point of the solid torus T. This closed loop defines an element in $\pi_1(T) = Z$. Adding all together for each intersection point, this is the linking number for two components. The linking matrix for a link in $S^1 \times D^2$ is the hermitian matrix for all pair of different components.

A framed link in $S^1 \times D^2$ is an oriented link with a framing on each component. A framing is just an integer. The self-linking number of a component of a framed link is the linking number of this component with a push off according to the framing. Hence the linking matrix of a framed link is a hermitian matrix with diagonal element is an element in $Z[Z]$ invariant under the involution i.e. an element of the general form $a_0 + \sum a_i(t^i + t^{-i})$.

Of course, the linking matrix depends on the choice of paths from the base point of the solid torus to base point of each component. But it does not depend on the bounding disks and which component one bounds.

Proposition 5 *Up to isomorphism, the linking matrix of a framed link is well defined.*

Proof: Think of $S^1 \times D^2$ as part of the boundary of $S^1 \times D^3$, then attach 2-handles to the framed link according to the framing. Then we have a 4-manifold with boundary and $\pi_1 = Z$. The linking matrix of the framed link is the intersection form on π_2 of this manifold in some basis. Therefore, this is well-defined up to isomorphism. □

For example, the Hopf link gives matrices $\begin{pmatrix} 0 & t^n \\ t^{-n} & 0 \end{pmatrix}$, where n is any integer.

Lemma 2 *Given any hermitian matrix, there is a framed link with linking matrix as it.*

Proof: Given an $n \times n$ hermitian matrix, choose n embedded disjoint disks D_i in $T = S^1 \times D^2$. If we modify the D_i by pushing a piece of D_i around a loop g and introduce a clasp with D_j, this changes the linking matrix by the symmetrization of the matrix with $\pm g$ in the i,j place if $i \neq j$. Any hermitian matrix over Z[J] can be realized by this operation except the diagonal elements. For the diagonal elements, frame the i-th component with a_0 of the i-th diagonal element $a_0 + \sum a_i(t^i + t^{-i})$. Then introduce homotopically trivial clasps of D_i with itself by pushing around t many times according to a_i. It is left as an exercise to check this gives the right linking matrix. □

Now we prove the classification theorem.

Proof: Let $S^1 \times D^3$ be a standard 1-handle. The boundary is $S^1 \times S^2$. Think of $S^1 \times S^2$ as 2 copies of the standard solid torus. Fix one copy T and a base point. Given the nonsingular hermitian form λ, let L be a link in T with linking matrix λ. Attaching 2-handles on the link according to the framing, we get a smooth, connected 4-manifold M_L with boundary. By nonsingularity of the form and Poincare duality, the boundary is a $Z[J]$ homology $S^1 \times S^2$. By [FQ], there is a 4-manifold homotopic to S^1 with boundary of this 3-manifold. Fill in such a 4-manifold to get a closed manifold. Let us check that the manifold built has the right properties: first it has $\pi_1 = Z$ by Van Kampen's theorem. To see it has the right intersection form, notice that there is a canonical basis for $\pi_2 M$ by capping off the core disks of the 2-handles with the bounding disks of the attaching circles.

For the weakly odd form, use the *-operation, we can get another one with opposite ks.

For the **Uniqueness.** Without loss of generality, assume M is the standard model built above. Let L be the link representing the hermitian form. Let $S^1 \longrightarrow N$ be a circle representing the generator of $\pi_1 N$. By general position, it is homotopic to an embedding. A neighbourhood of S^1 is a D^3-bundle over S^1. Since N is oriented, it is the trivial $S^1 \times D^3$. Now define a continous map from M_L to N as follows: identify $S^1 \times D^3$ in M_L with the regular neighbourhood of S^1 in N above. Note for the basis of $H_2(M_L, Z[J])$, there is a canonical choice. It consists of the immersed 2-spheres obtained by capping off the cores of the 2-handles with bounding disks for the attaching circles. Since M_L is homotopy equivalent to a wedge of one circle and some 2-spheres, choose a basis of $H_2(N, Z[J])$ and sending the basis to the basis according to h. This determines a map from M_L to N which is an embedding on $S^1 \times D^3$.

Remove the $S^1 \times D^3$ from M_L and its image in N, denote the manifolds left by M_0 and N_0. This gives a map from M_0 to N_0. There are two components of the boundary of M_0. Let ∂_0 be the boundary of the secondly removed $S^1 \times D^3$, and ∂_1 be the boundary of the 4-manifold homotopic to S^1 in the construction of the model. If the above map can homotoped rel ∂_0 to a π_1-negligible embedding, then the theorem can be finished as follows: fill in the 1-handle, we have an embedding from M_L to N. By π_1-negligibility, the completment of M_L in N must has $\pi_1 = Z$, Z[J]-homology 0. Therefore, it is a 4-manifold homotopic to S^1 with boundary the same as M_L. Extend the homeomorphism to M, we have a homoemorphism as desired.

Now it is sufficient to homotope the map to a π_1-negligible embedding f. Let $\{D_i\}$ be the core disks of the 2-handles in the model manifold, restrict the map to the cores, we have a map $f_0 : \{D_i\} \longrightarrow N_0$. Approximate this map by an immersion and resolve the framing problem as in the simply connected case. Then repeat the argument for the simply connected case exatly, we have a π_1-negligible embedding $f : M_0 \longrightarrow N_0$. □

4.6 Sum Decomposition

In this section, we will prove a sum decomposition theorem. This will be used in the next section to prove the uniqueness for nonorientable 4-manifolds with $\pi_1 = Z$.

A map $S^1 \longrightarrow M^4$ is always homotopic to an embedding by general position. The neighbourhood of it is a D^3-bundle over S^1, so it is either $S^1 \times D^3$ or a Mobius band times D^2, depending on if the loop is orientable or not. Now suppose there are embeddings of S^1 in both M and W on which ω_1 takes the same value.

Definition 8 *$M\sharp_{S^1}W$ is the 4-manifold obtained by deleting the interior of the disk bundle neighbourhood and identifying the boundary.*

Rem: This sum is not well-defined as claimed in [FQ] by giving local orientation. Actually, in $Diff(S^1 \times S^2) = Z_2^3$, and $Diff(S^1 \tilde{\times} S^2) = Z_2^2$ [KR], one Z_2 can not be controlled by local orientation. For example, it gives different manifolds for $RP^4\sharp_{S^1}RP^4$.

Suppose M has $\pi_1 = Z$ and $S^1 \subset M$ represents the generator, assume the inclusion $S^1 \subset W$ is injective on π_1.

Proposition 6 *(i) $\pi_1(M\sharp_{S^1}W) = \pi_1 W$, denote it by π.*

(ii) $H_2(M\sharp_{S^1}W; Z[\pi]) = H_2(M; Z[J]) \otimes_{Z[J]} Z[\pi] \oplus H_2(W; Z[\pi])$

(iii) The intersection form of the sum is the form on M tensored up to $Z[\pi]$, plus the form on W.

Now we have the following

Theorem 9 *Suppose M is a closed locally oriented 4-manifold with $\pi_1 = J$, W has good fundamental group π and no 2-torsion, $J \longrightarrow \pi$ is injective, and ω_1 of the two manifolds agrees on J, suppose that*

$$H_2(M; Z[J]) \otimes_{Z[J]} Z[\pi] \longrightarrow H_2(W; Z\pi)$$

is a $Z[\pi]$ monomorphism which preserves λ and $\tilde{\mu}$, and either $\omega_2 = 0$ or ω_2 does not vanish on the subspace perpendicular to the image, then there is a decomposition $W = M\sharp_{S^1}W'$ realizing the given homomorphism to $\pi_2 W$. If $\omega_2 \neq 0$ does vanish on the perpendicular subspace, then exactly one of W or ${}^*W$ decomposes.

Before we prove the theorem, we need two more lemmas.

Lemma 3 *Suppose that $W \cong M\sharp W_1 \cong M\sharp W_2$ induce the same decomposition of $\pi_2 W$ and W is compact and $\pi_1 W$ is good, if M is a closed 1-connected even manifold, then $W_1 \cong W_2$.*

Proof: Define $\hat{W}$ by connected sum with a copy of -M, so $\hat{W} \cong \hat{M}\sharp W_1 \cong \hat{M}\sharp W_2$ with $\hat{M} = M\sharp(-M)$. The form of $\hat{M}$ is even, indefinite, and has signature 0, so it is isomorphic to $k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for some k. Hence $\hat{M} = k(S^2 \times S^2)$. Denote the 2-spheres of two factors by $\{A_i\}$, and $\{B_i\}$.

Let $h_j : \hat{W} \cong \hat{M}\sharp W_j$ for j=1, 2 denote the given homeomorphism, so that $h_1 h_2^{-1}$ preserves the decomposition of the form and is identity on the part coming from $\hat{M}$. Construct a 5-manifold Z by starting with $\hat{W} \times I$, adding 3-handles on the sphere $h_1^{-1}A_i \subset \hat{W} \times \{0\}$, and 3-handles on the sphere $h_2^{-1}B_i \subset \hat{W} \times \{1\}$. The boundary of Z is the union of three pieces: W_1 connected sum the manifold obtained by surgery on $\cup_i A_i \subset \hat{M}$, W_2 connected sum the manifold obtained by surgery on $\cup_i B_i \subset \hat{M}$ and $\partial W \times I$. Since $\hat{M}$ is reduced to a homotopy sphere (thus a sphere) by surgery on either A_i or B_i, the first two pieces are W_1 and W_2. Then Z is an s-cobordism rel ∂ from W_1 to W_2. Since $\pi_1 W$ is good, the s-cobordism implies the two ends are homoemorphic. □

Lemma 4 *Let W be a compact cobordism between connected boundaries ∂_0 and ∂_1, and $\pi_1(W, \partial_0) = 0$, $\pi_1(W, \partial_1) = 0$, then after connected sum with closed 1-connected even 4-manifolds, (W, ∂_0) has a handlebody structures with only 2-handles.*

Proof: If W is not smooth, by connected sum with E_8, we may assume that ks=0. Then by connected sum with $S^2 \times S^2$'s, we may assume it is smooth, hence there is a handlebody structure. Since ∂_0, and ∂_1 are connected, there is a handlebody structure of W without 0- and 4-handles. Now we will show that 1- and 3-handles can be changed into 2-handles by connected sum with $S^2 \times S^2$'s.

Consider a 1-handle, attached to ∂_0. Since $\pi_1(W, \partial_0) = 0$, so the core of the 1-handle is homotopic relative to the ends into ∂_0. Approximate the homotopy by a framed immersed 2-disk which is standard near the core of the handles, and push selfintersections off by finger move. This gives a framed embeded disk whose boundary is the union of the handle core and an arc on ∂_0. Let B denote a 3-disk neighbourhood of the arc in ∂_0, then a collar on B union with the 1-handle is isomorphic to $S^1 \times D^3$. The connected sum operation with $S^2 \times S^2$ is the same as surgery on the S^1, therefore by such a connected sum we can replace the 1-handle on $B \times I$ with a copy of $D^2 \times S^2$. This can be regarded as a 2-handle added on $B \times I$, so the original 1-handle has been replaced by a 2-handle. A 3-handle can be considered as a 1-handle attached to ∂_1, so the same argument can be used to convert all 3-handles into 2-handles. □

Proof: Denote by M_S the complement of the open disk bundle of the embedding $S^1 \subset M$, then M_S is homotopically equivalent to a wedge of 2-spheres on a circle. Since $H_2(M; Z[J])$ is free, so the above data define a map $f : M_S \longrightarrow W$. Since $\pi_1(M_S) = Z$, let $S^1 \subset M_S$ be a generating circle with disk bundle E. Approximate f by an immersion, by general position, $f(S^1)$ is an embedding and misses the image of the wedge of 2-spheres in W. Hence we may assume that f is homotopic to a map which is a homeomorphism on E and takes $M_S \backslash intE$ into the complement of the image of E. Let $M_{S,E} = M_S \backslash intE$, $W_E = W \backslash int(imE)$, denote this map $(M_{S,E}, \partial E) \longrightarrow (W_E, \partial(imE))$ by f, too. Now if we can homotope f $rel \partial E$ to a π_1-negligible embedding, then fill in a D^3-bundle over S^1 for E, it follows that $W = M \sharp_{S^1} W'$ and $\pi_1 W' = \pi_1 W$.

By the lemma above, it is sufficient to find an embedding for the stabilized manifolds rel ∂E. Let V be a closed even 1-connected 4-manifold, and assume the map obtained by connected sum with the identity $V \sharp M_{S,E} \longrightarrow V \sharp W_E$ is homotopic to a π_1-negiligible embedding rel ∂E. This gives a second embedding of V_0 in $V \sharp W_E$, and therefore a decomposition $V \sharp W_E \cong V \sharp W'_E$ with V represents the same summand of the form. The the above lemma shows that the canonical homotopy equivalence $W_E \cong W'_E$ is homotopic rel ∂imE to a homeomorphism. But W'_E has $M_{S,E}$ embeded π_1-negligibly in it, therefore we get an embedding of $M_{S,E}$ in W_E. After addition of a closed 1-connected manifolds with even form, we may assume that $(M_{S,E}, \partial E)$ has a handlebody structure with only 2-handles.

Denote the core two disks of the handles by $D = \{D_i\}$, by lemma 1.3 [FQ], f is homotopic to an immersion which differs by rotations from the given one on the attaching region of handles. Let χ_i be the rotation on $D_i \cap \partial E$, and $A_i = D_i \cup f(D_i)$. Then A_i defines an immersed 2-sphere with possibly nontrivial normal bundle. The coefficients of 1 (with $Z[\pi_1 W]$ coefficient) of the selfintersection form of those immersions satisfy

$$2\mu_1(A_i) + \chi_i = \lambda_1(A_i, A_i).$$

But since that f preserves λ and $\tilde{\mu}$ implies that $\lambda_1(A_i, A_i)$ are even, so χ_i are even. Now this can be changed by twisting inside $f(D_i)$, so we can arrange it to be 0. This gives an immersion of $M_{S,E}$ extending the embeddind of ∂E. Since the intersection form of M is nonsingular, therefore all D_i has a dual class α_i such that $\lambda(D_i, \alpha_j) = \delta_{ij}$.

By section 3.2, there is an obstruction km(f) for this to be homotopic to an embedding. If $\omega_2 = 0$, then all dual 2-spheres can be framed, hence km(f)=0. If ω_2 is not 0 on the perpendicular subspace of the image, by adding one class to the dual, we can arrange km(f)=0. Therefore, in both cases we have an embedding. If ω_2 does vanish on the perpendicular subspace, then there is embedding in one of W or $*W$. □

4.7 Classification for nonorientable $\pi_1 = Z$

In this section, we carry out the classification in the nonorientable case. As in the orientable case, π_2 is still free. But the intersection form is defined using a local orientation at the base point. Therefore, the intersection form is a ω_1-hermitian nonsingular form. The classification is not as explicit as in the orientable case. The method in this section works as well for the orientable case.

Theorem 10 *(1).* **Existence:** *Suppose (H, λ) is a nonsingular ω_1-hermitian form on a finitely generated free Z[J]-module, $k \in Z_2$, and if λ is even, assume $k \cong [\lambda] \in L_4(Z, -)$, then there is a nonorientable closed 4-manifold with $\pi_1 = J$, intersection form λ and Kirby-Siebenmann invariant k.*

(2). **Uniqueness:** *Suppose M, N both are closed, not orientable but oriented locally, $h : H_2(M, Z[J]) \longrightarrow H_2(N, Z[J])$ is an isomorphism preserving the intersection form and ksM=ksN, then there is a homeomorphism $f : M \longrightarrow N$ which induces the identification of fundamental group, preserves the local orientation and $f_* = h$.*

Proof: For the existence, if λ is even it admits a unique quadratic refinement. By [Wa], $L_4(1) = Z \longrightarrow L_4(Z^-) = Z_2$ is onto. Then it follows that λ is stably isomorphic to $(E_8 \otimes_Z \Lambda) \oplus \cdots \otimes (E_8 \otimes_Z \Lambda)$. The latter is the intersection form of $W = S^1 \tilde{\times} S^3 \sharp E_8 \cdots \sharp E_8$. Thus there are r, s such that $\lambda \oplus H(\Lambda)^r$ is the intersection form of $W \sharp s(S^2 \times S^2)$, where $H(\Lambda)$ is the hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. However $H(\Lambda)^r$

can be realized in $W\sharp s(S^2 \times S^2)$ by a topological embedding of $\sharp_r(S^2 \times S^2)\backslash D^4$. By surgering this out, λ can be realized by a closed 4-manifold.

Assume λ is odd. By [MR], it follows that $L^0(1) = Z \longrightarrow L^0(Z^-) = Z_2$ is onto. Thus λ is stably isomorphic to $\oplus p(1) \oplus q(-1)$. But in this stablization, metabolic forms are used instead of hyperbolic forms. But for odd forms, stabilization with metabolic forms is the same as hyperbolic forms (see the following lemma), therefore there are r, s such that $\lambda \oplus H(\Lambda)^r$ is the intersection form of $W = S^1 \tilde{\times} S^3 \sharp pCP^2 \sharp q\overline{CP^2}$. But again, $H(\Lambda)^r$ can be realized by a topological embedding of $\sharp_r(S^2 \times S^2)\backslash D^4$. By surgering this out, the desired manifold is obtained.

Uniqueness: Given h as in (2). Regarding $h : H_2(M, Z[J]) \longrightarrow H_2(N, Z[J])$ as an injective. By section 4.6, there is a decomposition of $N = M\sharp_{S^1}P$ or $*N = M\sharp_{S^1}P$ realizing the injection. P_S is a manifold with $\pi_1 = Z$ and homotopy type of $S^1 \tilde{\times} D^3$. By [FQ], it is homeomorphic to it. Therefore, either $N = M$ or $*N = M$. Since * changes the ks if it is a different manifold, we conclude there is homoemorphism to induce h. □

To tie the loose end in the existence part, let us prove the following

Lemma 5 *If λ is odd, then stabilization with* $\begin{pmatrix} 0 & I \\ I & A \end{pmatrix}$ *is the same as* $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Proof: By writing $A = B + \bar{B} + D$, where $D = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}$, $\epsilon = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, we have the following identity:

$$\begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \cdot \begin{pmatrix} 0 & I \\ I & A \end{pmatrix} \cdot \begin{pmatrix} I & -\bar{B} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & D \end{pmatrix}$$

Hence it is sufficient to prove that stably

$$\lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cong \lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since λ is odd, we can find an element v such that $\lambda(v, v) = x + \bar{x} + 1$ for some x. But after stabilization, there is some $w \perp v$ such that $\lambda(w, w) = -x - \bar{x}$. Hence there is some element u such that $\lambda(u,u) = 1$. Take the orthogonal complement of u as C, then $\lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = <1> \oplus C$. But

$$<1> \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong <1> \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Geometrically, this is the identity: $CP^2\sharp(S^2 \times S^2) = CP^2\sharp\overline{CP^2}\sharp CP^2$. Now

$$\lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong C \oplus <1> \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\cong C \oplus <1> \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the lemma follows. □

The existence part of the nonorientable theorem can also be proved by a construction similiar to the orientable case. We have the following theorem which is a generalization of theorem 11.6 in [FQ].

Theorem 11 *If N is a Z[J]-homology $S^1 \tilde{\times} S^3$, then there is a 4-manifold homotopic to S^1 with boundary N.*

Proof: As in the proof of theoren 11.6 [FQ], it is sufficient to find a normal map to a Poincare pair (X,N), where X is a Poincare complex with homotopy type S^1. Passing the infinite cyclic cover, the manifolds are orientable with boundary. Then we can find normal maps there rel boundary.

Chapter 5

Classification of self homeomorphism

In this chapter we study the classification of homeomorphisms of a 4-manifold with fundamental group Z up to pseudoisotopy. An automorphism of the intersection form on π_2 which preserves the intersection form will be called an *isometry* on π_2. Given an isometry on π_2, there is a homeomorphism to induce this isometry. It has been shown [Q1] that for simply connected 4-manifolds, it is unique up to isotopy.

5.1 Uniqueness of sum decompositions

In chapter 4, it has been shown that under some conditions, an algebraic decomposition of π_2 correspondes to a sum decomposition. In this section, the uniqueness of such a decomposition is studied.

Theorem 12 *Suppose M is a closed 4-manifold with fundamental group Z, and W has good fundamental group. Suppose $h_1 : W \cong M\sharp_{S^1} W_1$ and $h_2 : W \cong M\sharp_{S^1} W_2$ are two decompositions inducing the same decomposition of π_2. If $\pi_1 W$ is good, and has no 2-torsion, and W is weakly even,*

then the decomposition are pseudoisotopic.

Rem: For the simply connected case, the above theorem appears in [FQ].

Two decompositions are pseudoisotopic if there is a pseudoisotopy $H : W \times I \longrightarrow W \times I$ of W from identity to a homeomorphsim g so that $h_2 g h_1^{-1}$ is the identity on $M_S = M \backslash \mathcal{N}(S^1)$.

Proof: Let M_S and $M_{S,E}$ as in section 4.6. Then h_1, h_2 determine maps from M_S to W. Since the homomorphism on π_2 determines the map up to homotopy, so if the decomposition induce the same decomposition of $\pi_2 W$, then the maps are homotopic rel ∂E.

Let the homotopy be

$$g : (M_{S,E} \times I, M_{S,E} \times \{0,1\} \cup \partial E \times I) \longrightarrow (W_E \times I, W_E \times \{0,1\} \cup \partial imE \times I).$$

Let $(X, \partial_0 X) = (M_{S,E} \times I, M_{S,E} \times \{0,1\} \cup \partial E \times I)$. Now if we can prove that there is a π_1-negligible embedding g' wich agree with g on $\partial_0 X$, then this is the pseudoisitopy we need. By filling in $E \times I$, we have an embedding of $M_S \times I \longrightarrow W \times I$. The π_1-negligibility and duality imply that the conplement of the interior of $g'(M_S \times I)$ is an s-cobordism. Since W has good π_1, then the s-cobordism implies this has a product structure. So there is a pseudoisotopy.

First we arrange the $(X, \partial_0 X)$ to have a handlebody structure with only 3-handles. Then construct dual 2-spheres for the image of the 3-handles. If the core disks of the handles are denoted by $D = \{D_i^3\}$, approximate them to be in general position and transverse to each other. Then the intersection are circles and arcs of double points. By theorem 3.3, we have the result since the obstruction vanishes in this case. □

5.2 Classification for homeomorphisms

Using the uniqueness of sum decomposition, we are able to classify homeomorphisms that induce the same isometry on π_2.

Theorem 13 *Let M^4 be a closed 4-manifolds with $\pi_1 = Z$, and f, g be two homoemorphisms that induce the same isometry on π_2, if M is weakly even, then they are pseudoisotopic. If it is weakly odd, then there are at most two equivalent classes under pseudoisotopy.*

Proof: Regarding M as $M\sharp_{S^1}(S^1 \times S^3)$, if M is weakly even, then by theorem 5.1, there is a pseudoisotopy of the identity of M to a homeomorphism h so that $f \cdot h \cdot g^{-1}$ is the identity on M_S. Now this pseudoisotopy can be extended to $M \times I$ by the uniqueness of $S^1 \times D^4$.

If M is weakly odd, then do the same thing as in section 5.1. Now we have an obstruction. But by section 3.3, this is a Z_2 obstruction. If the obstruction does not vanishes, then stack any two together, the obstruction vanishes. Hence there are at most two classes.

Remark: In [Q1], F. Quinn proved that pseudoisotopy and isotopy are the same for the simply connected 4-manifolds. But in general there is an obstruction for the pseudoisotopy to be isotopy for $\pi_1 = Z$.

5.3 Obstruction for pseudoisotopy

In this section, we will try to produce two homeomorphisms that induces the same isometry on π_2 and are not pseudoisotopic. It is clear that it is sufficient to consider that if $g : M \longrightarrow M$ is a homeomorphism such that $g_* = id$, whether or not that g is pseudoisotopic to identity. Since if $f_* = g_*$, then let $h = f^{-1} \cdot g$, then $h_* = id$. If h is pseudoisotopic to identity, by composing with $f \times id$, f is pseudoisotopic to g.

Given M, we are interested to find a homeomorphism of M such that $g_* = id$, but g is not pseudoisotopic to identity. If M is simply connected, or $\pi_1 = Z$ and weakly even, this is impossible. Hence, we asssume M is weakly odd. Remove 2 solid tori $S^1 \times D^3$ as in section 4.6, denote the manifold left by $M_{S,E}$. Let $Id \times I : M_{S,E} \times I \longrightarrow M_E \times I$. Let $\{D_i^2\} = D^2$ be the core 2-disks of $M_{S,E}$ after stabilization, and $\{D_i^3\} = \{D_i^2 \times I\}$. In the following, we will use $M_{S,E}$, and M_E to denote the stabilized manifolds, too. We are going to change the identity $M_{S,E} \subset M_E$ to an embedding $g : M_{S,E} \longrightarrow M_E$ such that $g_* = id$, but g is not pseudoisotopic to identity.

Without loss of generality, assume there is only one disk. Do a finger move to introduce 2 Hopf link self intersections with group element 1. Do another finger move to introduce 2 unkotted and unlinked circles H and H' with group element 1. Arrange the second finger move to be carried out along an arc in $M_E \times \{1/2\}$ and is symmetric about this slice. Choosing bounding disks for H and H' which has two clasps which is also done symmetrically about $M_E \times \{1/2\}$, one clasp above and one clasp below. Choose a band that will be used to split H and H' lying in $M_E \times \{1/2\}$ and thickened in the I direction. The bounding disks for H and H' are not standard. Hence, it is not the reverse of the finger move used to create H and H'. By the lemma 1, the framing is correct. Since $\pi_1 M$ is good, there is a π_1-negiligible embeded Whitney disk in $M_E \times \{1/2\}$. Using this Whitney disk we obtain an embedding that is symmetric with respect to the slice $M_E \times \{1/2\}$ (with one Hopf link above and one below representing 1) and in the slice it is an embeded 2-disk. Restricted to $D^2 \times [0, 1/2] \longrightarrow M \times [0, 1/2]$ producing an embedding g with $km(g) = 1 \in H_1(M; Z_2)$. Note the Whitney disk construction above is supported in a neighbourhood of $M \times \{1/2\}$ and it is exactly a 4-manifold Whitney disk problem crossed with R. It is also clear that the embedding produced is homotopic to identity on the 2-skeleton.

Now there is an embedding $g : M_{S,E} \longrightarrow M_E$. Fill in E, we have an embedding of M_S to M. By the homotopy condition, what left is $S^1 \times D^3$. Extend the embedding to this piece, obtain the embedding g as desired. Since it identifies π_1 and is identity on π_2, therefore, it is identity on $H_3(M, \Lambda)$ [LP]. Hence $g_* = id$.

Now we have a map from $M \times I$ to itself which are identity and g on the two ends. This map can not be a homotopy beween identity and g by the following theorem. The obstruction to change this map to an embedding rel boundary depends on the map itself and $H_i(M; Z[J])$. The homeomorphism g above is built in 3-stages. Over the 1-skeleton, it is by general position. Over the 2-skeleton, the obstruction was produced. Finally, an extension over a solid torus by the uniqueness of solid torus. The last extension did not affect the obstruction, but the first stage may change something. Therefore, in this construction, we have to analyze if the obstruction can be changed by th first filling in. This is a rather untractible problem.

Conjecture: There is a homeomorphism $g : CP^2 \sharp S^1 \times S^3 \longrightarrow CP^2 \sharp S^1 \times S^3$ such that $g_* = id$ and g is not pseudoisotopic to identity.

While it is very difficult to prove this geometrically, we will reduce this to a purely homotopy theoretical problem.

Theorem 14 *Two homeomorphisms of a closed orientable 4-manifold M with $\pi_1 = Z$ are homotopic iff they are pseudoisotopic.*

Proof: It is only neccesaary to prove the "only if" part.

Let $H : M \times I \longrightarrow M \times I$ be a homotopy between f and g, then H defines an element of the structur set $S_{TOP}(M \times I, \partial)$. To show that f and g are pseudoisotopic, it is sufficient to show there is a homotopy which is 0 in $S_{TOP}(M \times I, \partial)$.

Given a closed orientable 4-manifold M^4 with $\pi_1 = Z$, there is an exact sequence

$$0 \longrightarrow S_{TOP}(M \times I; \partial) \longrightarrow NM_{TOP}(M \times I; \partial) \longrightarrow L_5(Z).$$

The left 0 is because of $L_6(Z)$ acts nontrivially on $S_{TOP}(M \times I; \partial)$. $L_5(Z) = Z$ and $NM_{TOP}(M \times I, \partial) = Z \oplus Z_2$. The Z-factor in NM_{TOP} cancells the one in $L_5(Z)$. Therefore,

$$0 \longrightarrow S_{TOP}(M \times I, \partial) \xrightarrow{\eta} Z_2 \longrightarrow 0$$

gives $S_{TOP}(M \times I, \partial) = Z_2$. And the map η is a homomorphism.

Proposition 7 *There is a homotopy equivalence $h : M \times I \longrightarrow M \times I$ such that $h\,|_{M \times \partial I}$ is identity and $\eta(h) \neq 0$.*

Proof: In [Sh2], such a homotopy equivalence was constructed for the smooth case. This also gives an element for the topological case.

To finish the proof of the theorem, if $\eta(H) = 0$, it is done. If $\eta(H) \neq 0$, let $H' = h \cdot H$, then $\eta(H') = 0$ which gives us a pseudoisotopy.

By the above theorem, it is possible to study along the line [CH] by homotopy method.

Definition 9 *(Pinching operation) Let $c : M \longrightarrow M \vee S^4$ be the map which pinches off a top cell, and let $\tau : S^4 \longrightarrow M$ be a map. Define $< \tau >$ to be the composition $(1, \tau) \cdot c : M \longrightarrow M \vee S^4 \longrightarrow M$. This is a map of M which is homotopic to identity on the 2-skeleton.*

This construction defines a map from $\pi_4 M$ to the maps that induces the identity on homology with $Z[\pi_1]$ coefficients.

Lemma 6 *This homomorphism is onto.*

Proof: Analogous to [Q1].

The next step in this approach is to determine which map is homotopic to identity. This is in general very complicated. In the following, we will only study the manifold $CP^2 \sharp S^1 \times S^3$ again. Let $\alpha : S^3 \longrightarrow M$ be the identity map of S^3 into the boundary of the 4-ball deleted for the connected sum. $\sum \cdot \eta : S^4 \longrightarrow S^3$ be the suspension of the Hopf map. Then $\beta = \alpha \cdot \sum \cdot \eta$ defines an element in $\pi_4 M$.

Lemma 7 *α is not homotopically trivial. Actually, it is not even stably trivial.*

Proof: Using the stable homotopy theory.

Lemma 8 *Let $< \beta >: M \longrightarrow M \vee S^4 \longrightarrow M$ be the pinching map corresponding to β. If M is odd, then the normal invariant of this map is trivial.*

Proof: Using the characteristic variety theorem of [Su].

If we can prove this map $< \beta >$ is not homotopic to identity, then this will provide us the g as follows: since the normal invariant of $< \beta >$ is 0, hence there is a homeomorphism g such that $< \beta > \cong g \cdot id = g$. $< \beta >_* = id$ imply that $g_* = id$. But if $< \beta >$ is not homotopic to identity, neither is g. Unfortunately, it is still unkown if $< \beta >$ is homotopic to identity or not.

Chapter 6

Applications and Questions

6.1 Homotopy type

From chapter 4, it follows

Corollary 1 *The homotopy type of a closed 4-manifold with fundamental group 0 or Z is determined by the intersection form on π_2.*

In general, it is difficult to find a complete set of invariants to easily determine the homotopy type. For the simply connected case, this is obtained earlier by J.H.C Whitehead. For the fundamental group Z, Peter Teichner proved this directly[Te2].

6.2 Exotic smooth structures

For a smooth 4-manifold, we can use the classification to determine if there is an exotic smooth structure. The following observation is obvious: if there is an automorphism of the intersection form that can not be induced by a diffeomorphism, but can be induced by a homeomorphism, then the smooth structure is not unique. By the classification theorem in chapter 4, for 4-manifolds with $\pi_1 = 0, Z$ any automorphism of the intersection form on π_2 can be induced by a homeomorphism. By [Sh2], there are such automorphism for $S^1 \times S^3 \sharp S^2 \times S^2$ which are not induced by diffeomorphisms. It follows

Corollary 2 *There are exotic smooth structures on* $S^1 \times S^3 \sharp S^2 \times S^2$.

In [Sh2], a general method is described to determined when an automorphism can be induced by a diffeomorphism. Here we are interested in when an automorphism can not induced by a diffeomorphism. For $\pi_1 = 0$, some are known by S. Donaldson. In [Sh2], there are more examples for the $\pi_1 = Z$ case. But we do not know in general how well this works.

6.3 A final question

The classification of chapter 4 was based on the intersection form on π_2. It would be much better if this can be replaced by the intersection form on H_2. Since any free $Z[J]$-module comes from a Z-module by tensoring with $Z[J]$, it is only a question for the hermitian form. Using the determinant invariant, it is not difficult to see that not any quadratic form on a free Z[J]-module comes from an integral form, but it seems difficult for hermitian forms. If every hermitian form on a Z[J]-module is an extension of an integral form, then 4-manifolds with $\pi_1 = Z$ are much easier to understand. They will be always a connected sum of a simply connected 4-manifold with $S^1 \times S^3$ or $S^1 \tilde{\times} S^3$. Somehow, we believe that this should be true. Up to now, we are not aware of any 4-manifold with $\pi_1 = Z$ that can not be written as such a connected sum.

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