Non-Abelian quantum Hall states and their quasiparticles: From the pattern of zeros to vertex algebra

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In the pattern-of-zeros approach to quantum Hall states, a set of data $\{n;m;S_a|a=1,\dots,n;n,m,S_a\in\mathbb{N}\}$ (called the pattern of zeros) is introduced to characterize a quantum Hall wave function. In this paper we find sufficient conditions on the pattern of zeros so that the data correspond to a valid wave function. Some times, a set of data $\{n;m;S_a\}$ corresponds to a unique quantum Hall state, while other times, a set of data corresponds to several different quantum Hall states. So in the latter cases, the pattern of zeros alone does not completely characterize the quantum Hall states. In this paper, we find that the following expanded set of data $\{n;m;S_a;c|a=1,\dots,n;n,m,S_a\in\mathbb{N};c\in\mathbb{R}\}$ provides a more complete characterization of quantum Hall states. Each expanded set of data completely characterizes a unique quantum Hall state, at least for the examples discussed in this paper. The result is obtained by combining the pattern of zeros and Z_n simple-current vertex algebra which describes a large class of Abelian and non-Abelian quantum Hall states $\Phi^{\text{sc}}_{Z_n}$. The more complete characterization in terms of $\{n;m;S_a;c\}$ allows us to obtain more topological properties of those states, which include the central charge c of edge states, the scaling dimensions and the statistics of quasiparticle excitations.

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I. INTRODUCTION

Materials can have many different forms, which is partially due to the very rich ways in which atoms and electrons can organize. The different organizations correspond to different phases of matter (or states of matter). It is very important for physicists to understand these different states of matter and the phase transitions between them. At zero temperature, the phases are described by the ground-state wave functions, which are complex wave functions $\Phi(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_N)$ with $N\to\infty$ variables. So mathematically, to describe zero-temperature phases, we need to characterize and classify the ground-state wave functions with ∞ variables, which is a very challenging mathematical problem.

For a long time we believe that all states of matter and all phase transitions between them are characterized by their broken symmetries and the associated order parameters.[1] A general theory for phases and phase transitions is developed based on this symmetry breaking picture. So within the paradigm of symmetry breaking, a many-body wave function is characterized by its symmetry properties. Landau's symmetry breaking theory is a very successful theory and has dominated the theory of phases and phase transitions until the discovery of fractional quantum Hall (FQH) effect.[2,3]

FQH states cannot be described by symmetry breaking since different FQH states have exactly the same symmetry. So different FQH states must contain a new kind of order. The new order is called topological order[4–6] and the associated phase called topological phase because their characteristic universal properties (such as the ground states degeneracy on a torus[4]) are invariant under *any* small perturbations of the system. Unlike symmetry-breaking phases described by local order parameters, a topological phase is characterized by a pattern of long-range quantum entanglement.[7–9] In Ref. 10, the non-Abelian Berry phases for the degenerate ground states are introduced to systematically characterize and classify topological orders in FQH states (as well as other topologically ordered states). In this paper, we further develop another systematic characterization of the topological orders in FQH states based on the pattern of zeros approach.[11,12]

In the strong magnetic field limit, a FQH wave function with filling factor $\nu<1$ is an antisymmetric holomorphic polynomial of complex coordinates $\{z_i=x_i+iy_i\}$ [except for a common factor that depends on geometry: say, a Gaussian factor $\exp(\Sigma_i\frac{|z_i|^2}{4})$ for a planar geometry]. After factoring out an antisymmetric factor of $\Pi_{i<j}(z_i-z_j)$, we can describe a quantum Hall state by a symmetric polynomial $\Phi(z_1,\dots,z_N)$ in the $N\to\infty$ limit.[11] So the characterization and classification of long-range quantum entanglements in FQH states become a problem of characterizing and classifying symmetric polynomials with infinite variables.

In a recent series of work,[11–13] the pattern of zeros is introduced to characterize and classify symmetric polynomials of infinite variables. The pattern of zeros is described by a sequence of integers $\{S_a|a=1,2,\dots\}$, where S_a is the lowest order of zeros of the symmetric polynomial when we fuse a different variables together. The data $\{S_a|a=1,2,\dots\}$ can be further compactified into a finite set $\{n;m;S_a|a=1,2,\dots,n;n,m\in\mathbb{N}\}$ for n-cluster quantum Hall states. Here $\mathbb{N}=\{0,1,2,\dots\}$ is the set of non-negative integers. It has been shown[11,12] that all known one-component Abelian and non-Abelian quantum Hall states can be (partially) characterized by pattern of zeros. It is also shown[11,12] that, for any given pattern of zeros $\{S_a\}$, we can construct an ideal local Hamiltonian[14–18] $H_{\{S_a\}}$ such that the FQH state with the pattern of zeros is a zero energy ground state of the Hamiltonian.

We would like to point out that, strictly speaking, a FQH state must be a state with a finite energy gap. But in this paper, we will use the term more loosely. We will call one state a FQH state if it can be an zero energy state of an ideal Hamiltonian (whose interaction potential is a sum of δ functions and derivatives of δ functions). Such a FQH state is called an ideal FQH state. So our ideal FQH states may not be gapped.

The ideal FQH states has a nice property that they can be characterized by pattern of zeros or CFT. In addition to those ideal FQH wave functions, one can also construct FQH wave function through composite fermion approach.[19] Those composite fermion wave functions can be very low energies for Coulomb interaction, for example, for the $\nu=2/5$ state. However, the composite fermion wave functions are in general not the ideal FQH wave function defined above. We still do not know how to extract topological properties from the composite fermion wave functions. On the other hand, the ideal FQH wave functions introduced above are much easier to handle and we can indeed extract topological properties from the ideal FQH wave functions (provided that they are gapped).

Due to the length of this paper, in the following, we are going to summarize the issues that we are going to discuss in this paper. We will also summarize the main results that we obtain on those issues.

A. Sufficient conditions on pattern of zeros

Within the pattern-of-zero approach, two questions naturally arise: (1) Does any pattern of zeros, i.e., an arbitrary integer sequence $\{n;m;S_a\}$ corresponds to a symmetric polynomial $\Phi(z_1,\ldots,z_N)$? Are there any "illegal" patterns of zeros that do not correspond to any symmetric polynomial? (2) Given a "legal" pattern of zeros, can we construct a corresponding FQH many-body wave function? Is the FQH many-body wave function uniquely determined by the pattern of zeros?

For question (1), it turns out that the pattern of zeros must satisfy some consistent conditions[11,12] in order to describe an existing symmetric polynomial. In other words, some sequences $\{n;m;S_a\}$ do not correspond to any symmetric polynomials. However, Refs. 11 and 12 only obtain some necessary conditions on the pattern of zeros $\{n;m;S_a\}$. We still do not have a set of sufficient conditions on pattern of zeros that guarantee a pattern of zeros to correspond to an existing symmetric polynomial.

For question (2), right now, we do not have an efficient way to obtain corresponding FQH many-body wave function from a "legal" pattern of zeros. Further more, while some patterns of zeros can uniquely determine the FQH wave function, it is known that some other patterns of zeros cannot uniquely determine the FQH wave function: i.e., in those cases, two different FQH wave functions can have the same pattern of zeros.[11,20] This means that some patterns of zeros do not provide complete information to fully characterize FQH states. In this case it is important to expand the data of pattern of zeros to obtain a more complete characterization of FQH states.

We see that the above two questions are actually closely related. In this paper, we will try to address those questions. Motivated by the conformal field theory (CFT) construction of FQH wave functions,[21–25] we will try to use the patterns of zeros to define and construct vertex algebras (which are CFTs). Since the correlation function of the electron operator in the constructed vertex algebra gives us the FQH wave function, once the vertex algebra is obtained from a pattern of zeros, we effectively find the corresponding FQH wave function for the pattern of zeros. In this way, we establish the connection between the pattern of zeros and the FQH wave function through the vertex algebra.

In order for the correlation of electron operators in the vertex algebra to produce a single-valued electron wave function with respect to electron variables $\{z_1,\ldots,z_N\}$, electron operators need to satisfy a so-called "simple-current" property [see Eqs. (34) and (60)]. Also the vertex algebra need to satisfy the generalized Jacobi identity (GJI) which guarantees the associativity of the corresponding vertex algebra.[26] We find that only a certain set of patterns of zeros can give rise to simple-current vertex algebras that satisfy the GJI. So the GJIs in simple-current vertex algebras give us a set of sufficient conditions on a pattern of zeros so that this pattern of zeros does correspond to an existing symmetric polynomial.

In this paper, we first try to use the pattern of zeros $\{n;m;S_a\}$ to define a Z_n vertex algebra. From some of the GJI of the Z_n vertex algebra, we obtain more necessary conditions on the pattern of zeros $\{n;m;S_a\}$ than those obtained in Refs. 11 and 12 (see Sec. III). It is not clear if those conditions are actually sufficient or not.

Then, we try to use the pattern of zeros $\{n;m;S_a\}$ to define a Z_n simple-current vertex algebra. From the complete GJI of the Z_n simple-current vertex algebra, we obtain sufficient conditions on the pattern of zeros $\{n;m;S_a\}$ (see Sec. V).

B. How to expand the pattern-of-zeros data to completely characterize the topological order

If a pattern of zeros $\{n;m;S_a\}$ can uniquely describe the topological order in a quantum Hall ground state, then from such a quantitative description, we should be able to calculate the topological properties from the data $\{n;m;S_a\}$. Indeed, this can be done. First different types of quasiparticles can also be quantitatively described and labeled by a set of sequences $\{S_{\gamma;a}\}$ that can be determined from the pattern-of-zeros data $\{n;m;S_a\}$.[12] Those quantitative characterizations of the quantum Hall ground state and quasiparticles allow us to calculate the number of different quasiparticle types, quasiparticle charges, fusion algebra between the quasiparticles, and topological ground-state degeneracy on a Riemann surface of any genus.[12,13]

However, from the pattern-of-zeros data, $\{n;m;S_a\}$ and $\{S_{\gamma;a}\}$, we still do not know how to calculate the quasiparticle statistics and scaling dimensions, as well as the central charge c of the edge states. This difficulty is related to the fact that some patterns of zeros do not uniquely characterize a FQH state. Thus one cannot expect to calculate the topological properties of FQH state from the pattern-of-zeros data alone in those cases. We would like to point out that for a particular pattern of zeros for the Z_2 parafermion Moore-Read state, there is a successful calculation of the quasiparticle statistics from the pattern of zeros and thin cylinder limit.[27] But we do not know how to apply such an approach to more general pattern of zeros.

In this paper, we will try to solve this problem using a very different approach and for generic patterns of zeros. We first introduce a more complete characterization for FQH states in terms of a expanded data set: $\{n;m;S_a;c\}$. Then, we use the data set $\{n;m;S_a;c\}$ to define a so called Z_n simple-current vertex algebra. The Z_n simple-current vertex algebra contain a subalgebra, Virasoro algebra, generated by the energy-momentum tensor T and c is the central charge of the Virasoro algebra. It contains only n primary fields ψ_a, $a=0,1,\ldots,n-1$ of the Virasoro algebra, with a Z_n fusion rule $\psi_a\psi_b\sim\psi_{(a+b)\text{mod } n}$, $\psi_n=\psi_0$. Those ψ_a are called simple currents. The extra data c is the one of the structure constants of the Z_n simple-current vertex algebra. One may want to include all the structure constants $\{C_{ab}\}$ in the data set to have a complete characterization. But for the examples discussed in this paper, we find that data set $\{n;m;S_a;c\}$ already provides a complete characterization. So in this paper, we will use $\{n;m;S_a;c\}$ to characterize FQH states. If later we find that $\{n;m;S_a;c\}$ is not sufficient, we can always add additional data, such as C_{ab}. Every Z_n simple-current vertex algebra uniquely defines a FQH state, and the data $\{n;m;S_a;c\}$ that define a Z_n simple-current vertex algebra also completely characterizes a FQH state.

We would like to remark that although the data $\{n;m;S_a;c\}$ and the corresponding Z_n simple-current vertex algebras describe a large class of FQH states, they do not describe all FQH states. For example let $\Phi_{\mathcal{A}_i}$ be the FQH wave function described by a Z_{n_i} simple-current vertex algebra $\mathcal{A}_i$, $i=1,2$. Then, in general, the FQH state described by the product wave function $\Phi=\Phi_{\mathcal{A}_1}\Phi_{\mathcal{A}_2}$ cannot be described by a simple-current vertex algebra. Such a product state is described by the product vertex algebra $\mathcal{A}_1\otimes\mathcal{A}_2$, which is in general no longer a simple-current vertex algebra. So a more general FQH state should have the form

$$\Phi=\prod_i \Phi_{\mathcal{A}_i}. \tag{1}$$

The study in Refs. 11–13 reveal that many FQH states described by pattern of zeros have the following form:

$$\Psi(\{z_i\})=\prod_a \Phi_{Z_{n_a}^{(k_a)}}(\{z_i\}), \tag{2}$$

where $\Phi_{Z_{n_a}^{(k_a)}}(\{z_i\})$ is the wave function described by $Z_{n_a}^{(k_a)}$ parafermion vertex algebra.[13] The Z_n simple-current vertex algebra mentioned above is a natural generalization of the $Z_{n_a}^{(k_a)}$ parafermion vertex algebra, and Eq. (1) naturally generalizes Eq. (2). (Note that there are many Z_n simple-current vertex algebras even for a fixed n, so there are many different Z_n simple-current states.)

For the subclass of FQH states described by Z_n simple-current vertex algebra (which includes Virasoro algebra as an essential part), the quasiparticle statistics and scaling dimensions, as well as the central charge c of the edge states can be calculated from the data $\{n;m;S_a;c\}$. Certainly, we can also calculate the number of different quasiparticle types, quasiparticle charges, fusion algebra between the quasiparticles, and topological ground-state degeneracy on a Riemann surface of any genus.

Obviously, not every collection $\{n;m;S_a;c\}$ corresponds to a Z_n simple-current vertex algebra and a FQH state. GJIs of the Z_n simple-current vertex algebra generate the consistent conditions on the data set $\{n;m;S_a;c\}$. Only those data sets $\{n;m;S_a;c\}$ that satisfy the GJIs can describe a Z_n simple-current vertex algebra and FQH states.

C. Organization of the paper

This paper is organized as follows. In Sec. II, we review and extend the pattern-of-zeros approach to quantum Hall states. In Sec. III we use the pattern of zeros to define Z_n vertex algebra, and then use associativity conditions (i.e., the GJIs) of the vertex algebra to obtain extra conditions on the pattern of zeros that describe generic FQH states. In Sec. IV we list some numerical solutions of the pattern of zeros for the generic FQH states that also satisfy those extra consistent conditions found in Sec. III. In Sec. V we define and construct the Z_n simple-current vertex algebra from the pattern of zeros. We list the consistent conditions obtained from GJIs of Z_n simple-current vertex algebra. The detailed derivations of those consistent conditions are discussed in Appendixes D–F. The consistent conditions on the patterns of zeros that describe a Z_n simple-current vertex algebra are more restrictive than those for a generic Z_n vertex algebra. Some of the solutions of the Z_n simple-current pattern of zeros are listed in Sec. VII. In Sec. VI, we discuss how to represent quasiparticles in the Z_n simple-current vertex algebra, and to calculate the topological properties of quasiparticles from the Z_n simple-current pattern of zeros. In Sec. VII, we apply the vertex-algebra approach developed here to study some simple (but nontrivial) examples of FQH states, which include Z_n parafermion states (the Read-Rezayi states[17]), the Z_n simple-current FQH states of $Z_n|Z_n$ type, a Z_4 simple-current FQH state of $Z_4|Z_2$ type, etc.

II. PATTERN-OF-ZEROS APPROACH TO GENERIC FQH STATES

In this section, we will review how to use the pattern of zeros to characterize and classify different FQH states that have one component.[11–13] A discussion on two-component FQH states can be find in Ref. 28.

A. FQH wave functions and symmetric polynomials

Generally speaking, to classify generic complex wave functions $\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ is not even a well-defined problem. Fortunately, under a strong magnetic field, electrons are spin polarized in the lowest Landau level (LLL) when the electron filling fraction ν_e is less than 1. The wave function of a single electron in LLL (we set magnetic length $l_B=\sqrt{\hbar/eB}$ to be unity hereafter) is $\Psi_m(z)=z^m e^{-|z|^2/4}$ in a planar geometry.

m is the angular momentum of this single particle state. Thus the many-body wave function of *spin-polarized* electrons in the LLL should be

$$\Psi_e(z_1,\ldots,z_N)=\tilde{\Phi}_e(z_1,\ldots,z_N)\exp\left(-\sum_{i=1}^{N}\frac{|z_i|^2}{4}\right) \tag{3}$$

where $\tilde{\Phi}_e(\{z_i\})$ is an antisymmetric holomorphic polynomial of electron coordinates $\{z_i=x_i+iy_i\}$. The electron filling fraction ν_e is defined as

$$\nu_e=\lim_{N\to\infty}\frac{N}{N_\phi}=\lim_{N\to\infty}\frac{N^2}{2N_p}, \tag{4}$$

where N_ϕ is the total number of flux quanta piercing through the sample, and N_p is the total degree of polynomial $\tilde{\Phi}_e(\{z_i\})$. For FQH states $\nu_e<1$, we can extract a Jastraw factor $\Pi_{i<j}(z_i-z_j)$ and the remaining part

$$\Phi(z_1,\ldots,z_N)=\frac{\tilde{\Phi}_e(z_1,\ldots,z_N)}{\prod_{i<j}(z_i-z_j)} \tag{5}$$

would be a symmetric polynomial of $\{z_i\}$. We will concentrate on this symmetric polynomial to characterize and classify FQH states.

For the symmetric polynomial $\Phi(\{z_i\})$ we can also define a filling fraction ν in the same way as in Eq. (4), only N_p replaced by the total degree of bosonic polynomial $\Phi(\{z_i\})$. The electron filling fraction ν_e has the following relation with this *bosonic filling fraction* ν:

$$\nu_e=\frac{1}{1+\nu^{-1}}<1. \tag{6}$$

B. Fusion of a variables: The pattern of zeros

The pattern of zeros[11,12] is introduced to describe symmetric polynomials $\Phi(\{z_i\})$ through certain local properties, i.e., fusion of a different variables $z_1,\ldots,z_a$. More specifically, we bring these a variables together, viewing $z_{a+1},\ldots,z_N$ as fixed coordinates. By writing the a variables in the following manner $z_i=\lambda\xi_i+z^{(a)}$, $i=1,\ldots,a$, where $z^{(a)}=\frac{z_1+\cdots+z_a}{a}$ and $\Sigma_{i=1}^{a}\xi_a=0$, we can bring these a variables together by letting λ tend to zero. Then we can expand the polynomial $\Phi(\{z_i\})$ in powers of λ,

$$\begin{aligned}&\lim_{\lambda\to 0^+}\Phi(\lambda\xi_1+z^{(a)},\ldots,\lambda\xi_a+z^{(a)};z_{a+1},\ldots,z_N)\\ &\quad=\lambda^{S_a}P_{S_a}[z^{(a)},(\xi_1,\ldots,\xi_a);z_{a+1},\ldots,z_N]+O(\lambda^{S_a+1}).\end{aligned} \tag{7}$$

In other words, $\{S_a\}$ is the lowest order of zeros when we fuse a variables together. The pattern of zeros, by definition, is this sequence of integers $\{S_a\}$. In this paper, we will only consider the polynomials that satisfy a unique fusion condition: the fusion of a variables is unique, i.e., P_{S_a} in Eq. (7) has the same form except for an overall factor no matter how $\{\xi_i\}$ are chosen.

There are other equivalent descriptions of the pattern of zeros. One of them is the orbital description,

$$l_a=S_a-S_{a-1},\quad a=1,2,\ldots, \tag{8}$$

where $\{l_a\}$ labels the orbital angular momentum of the single-particle state occupied by the ath particle. Another is the occupation description in terms of a sequence of integers $\{n_l\}$,[29–32] denoting the number of particles occupying the orbital with angular momentum l.

C. Consistent conditions for the pattern of zeros

In this section, we will review and summarize the consistent conditions on $\{S_a\}$ derived in Refs. 11 and 12.

1. Translational invariance

A translational invariant wave function $\Phi(z_1,\ldots,z_N)=\Phi(z_1-z,\ldots,z_N-z)$ satisfies $\Phi(0,z_2,\ldots,z_N)\neq 0$. As a result we have $S_1=0$.

2. Symmetry condition

After we fuse a variables together to form an a-particle cluster (a-cluster), it is natural to ask: what happens when we fuse an a-cluster and another b-cluster together? Let $D_{a,b}$ be the order of zeros obtained by fusing an a-cluster and another b-cluster together. It satisfies $D_{a,b}=D_{b,a}\geqslant 0$. Since the final state is the same as fusing $a+b$ variables together, we find an one-to-one relation between the two sets of data $D_{a,b}$ and S_a,[11]

$$D_{a,b}=S_{a+b}-S_a-S_b,$$

$$S_a=\sum_{i=1}^{a-1}D_{i,1}. \tag{9}$$

Since $\Phi(\{z_i\})$ is a symmetric polynomial, it describes a state of bosonic particles seated at coordinates $\{z_i\}$. Thus the a-cluster seated at $z_i^{(a)}$ can also be regarded as a bosonic particle. The derived polynomial [see P_{S_a} in Eq. (7) as an example] should be symmetric with respect to interchange of two identical bosons seated at $z_1^{(a)}$ and $z_2^{(a)}$. When we fuse such two identical bosonic clusters $z_1^{(a)}$ and $z_2^{(a)}$ together, we have

$$\begin{aligned}\lim_{z_1^{(a)}\to z_2^{(b)}}P(z_1^{(a)},z_2^{(a)},\ldots)&=(z_1^{(a)}-z_2^{(a)})^{D_{a,a}}\tilde{P}\left(\frac{z_1^{(a)}+z_2^{(a)}}{2},\ldots\right)\\ &\quad+O[(z_1^{(a)}-z_2^{(a)})^{D_{a,a}+1}].\end{aligned} \tag{10}$$

This leads to the symmetry condition

$$D_{a,a}=\text{even}\Leftrightarrow S_{2a}=\text{even}. \tag{11}$$

3. Concave conditions

The first concave condition is the non-negativity of $D_{a,b}$,

$$D_{a,b}\geqslant 0\Leftrightarrow S_{a+b}\geqslant S_a+S_b. \tag{12}$$

It comes naturally from the fusion of two clusters.

When we fuse three clusters together, we find the total order of "off-particle" zeros to be

$$\Delta_3(a,b,c) = D_{a,b+c} - D_{a,b} - D_{a,c} \geqslant 0. \tag{13}$$

This gives the 2nd concave condition:

$$\Delta_3(a,b,c) = S_{a+b+c} + S_a + S_b + S_c - S_{a+b} - S_{a+c} - S_{b+c} \geqslant 0. \tag{14}$$

4. n-cluster condition

The above conditions, Eqs. (11), (12), and (14), have many solutions $\{S_a\}$. Many of those solutions has a "periodic" structure that the whole sequence $\{S_a\}$ can be determined from first a few terms,

$$S_{a+kn} = S_a + kS_n + mn\frac{k(k-1)}{2} + kma, \tag{15}$$

where

$$m \equiv D_{n,1}. \tag{16}$$

We will call such a pattern of zeros the one that satisfies an n-cluster condition. We see that, for an n-cluster sequence, only the first n terms, $S_2, \ldots, S_{n+1}$, are independent, and the whole sequence is determined by the first n terms.

To understand the physical meaning of the n-cluster condition, we note that Eq. (15) is equivalent to the following condition:

$$\Delta_3(kn,b,c) = 0 \quad \text{for any } k. \tag{17}$$

This means that a symmetric polynomial that satisfies the n-cluster condition has the following defining property: as a function of the n-cluster coordinate $z^{(n)}$, the derived polynomial $P(z^{(n)}, z_1^{(a)}, \cdots)$ has no off-particle zeros.

Under the n-cluster condition, we see that

$$D_{n,n} = nm = \text{even}. \tag{18}$$

We also note that the filling fraction is given by

$$\nu = \frac{n}{m} \tag{19}$$

since $S_a \to \frac{1}{2}\frac{m}{n}a^2$ as $a \to \infty$.

We like to mention that the cluster condition plays a very important role in the Jack polynomial approach to FQH.[31–33] However, in the pattern of zeros approach, the n-cluster condition only play a role of grouping and tabulating solutions of the consistent conditions. The solutions with larger n correspond to more complex wave functions which usually correspond to less stable FQH states. Later, we will discuss the relation between the pattern of zeros and CFT. We find that the solutions that do not satisfy the n-cluster condition (i.e., with $n=\infty$) correspond to irrational CFT, which may always correspond to gapless FQH states. The Jack polynomial approach and the pattern-of-zeros approach have some close relations. The Jack polynomials are special cases of the polynomials characterized by pattern of zeros.

5. Summary

To summarize, we see that the pattern of zeros for an n-cluster polynomial is described by a set of positive integers $\{n;m;S_2,\ldots,S_n\}$. Introducing $S_1=0$ and

$$S_{a+kn} = S_a + kS_n + mn\frac{k(k-1)}{2} + kma, \tag{20}$$

which define $S_{n+1}, S_{n+2}, \ldots,$ we find that the data $\{n;m;S_2,\ldots,S_n\}$ must satisfy

$$D_{a,b} = S_{a+b} - S_a - S_b \geqslant 0$$

$$D_{a,a} = \text{even}, \tag{21}$$

$$\Delta_3(a,b,c) = S_{a+b+c} + S_a + S_b + S_c - S_{a+b} - S_{a+c} - S_{b+c} \geqslant 0 \tag{22}$$

for all $a,b,c=1,2,3,\ldots$.

Conditions (21) and (22) are necessary conditions for a pattern of zeros to represent a symmetric polynomial. Although Eqs. (21) and (22) are very simple, they are quite restrictive and are quite close to be sufficient conditions. In fact if we add an additional condition

$$\Delta_3(a,b,c) = \text{even}, \tag{23}$$

the three conditions, (21)–(23), may even become sufficient conditions for a pattern of zeros to represent a symmetric polynomial.[11,12] However, this condition is too strong to include many valid symmetric polynomials such as Gaffnian,[34] a nontrivial Z_4 state discussed in detail in Sec. VII. We will obtain some additional conditions in Sec. III C, which combined with Eqs. (21) and (22) form a set of necessary and (potentially) sufficient conditions for a valid pattern of zeros.

D. Label the pattern of zeros by h_a^{sc}

In this section, we will introduce a new labeling scheme of the pattern of zeros. We can label the pattern of zeros in terms of

$$h_a^{\text{sc}} = S_a - \frac{aS_n}{n} + \frac{am}{2} - \frac{a^2 m}{2n}. \tag{24}$$

This labeling scheme is intimately connected to the vertex algebra approach that we will discuss later.

The n-cluster condition (20) of S_a implies that h_a^{sc} is periodic

$$h_0^{\text{sc}} = 0, \quad h_a^{\text{sc}} = h_{a+n}^{\text{sc}} \tag{25}$$

The two sets of data $\{n;m;S_2,\ldots,S_n\}$ and $\{n;m;h_1^{\text{sc}},\ldots,h_{n-1}^{\text{sc}}\}$ has a one-to-one correspondence since

$$S_a = h_a^{\text{sc}} - ah_1^{\text{sc}} + \frac{a(a-1)m}{2n}. \tag{26}$$

We can translate the conditions on $\{m;S_a\}$ to the equivalent conditions on $\{m;h_a^{\text{sc}}\}$. First, we have

$$2nS_a = 2nh_a^{\text{sc}} - 2nah_1^{\text{sc}} + a(a-1)m = 0 \bmod 2n,$$

$$nS_{2a} = nh_{2a}^{\mathrm{sc}} - 2nah_1^{\mathrm{sc}} + a(2a-1)m = 0 \bmod 2n,$$
$$m > 0, \quad mn = \text{even}, \tag{27}$$

$nS_{2n}=0 \bmod 2n$ in Eq. (27) leads to $2nh_1^{\mathrm{sc}}+m=0 \bmod 2$, from which we see that $2nh_1^{\mathrm{sc}}$ is an integer. From $2nh_a^{\mathrm{sc}}-a(2nh_1^{\mathrm{sc}})+a(a-1)m=$even integer, we see that $2nh_a^{\mathrm{sc}}$ are always integers. Also $2nh_{2a}^{\mathrm{sc}}$ are always even integers, and $2nh_{2a+1}^{\mathrm{sc}}$ are either all even or all odd. Since $h_n^{\mathrm{sc}}=0$, thus when $n=$odd, $2nh_a^{\mathrm{sc}}$ are all even. Only when $n=$even can $2nh_{2a+1}^{\mathrm{sc}}$ either be all even or all odd. When $m=$even, $2nh_{2a+1}^{\mathrm{sc}}$ are all even. When $m=$odd, $2nh_{2a+1}^{\mathrm{sc}}$ are all odd.

The two concave conditions become

$$h_{a+b}^{\mathrm{sc}} - h_a^{\mathrm{sc}} - h_b^{\mathrm{sc}} + \frac{abm}{n} = D_{ab} = \text{integer} \geqslant 0, \tag{28}$$

$$h_{a+b+c}^{\mathrm{sc}} - h_{a+b}^{\mathrm{sc}} - h_{b+c}^{\mathrm{sc}} - h_{a+c}^{\mathrm{sc}} + h_a^{\mathrm{sc}} + h_b^{\mathrm{sc}} + h_c^{\mathrm{sc}} = \Delta_3(a,b,c)$$
$$= \text{integer} \geqslant 0. \tag{29}$$

The valid data $\{n;m;h_1^{\mathrm{sc}},\ldots,h_{n-1}^{\mathrm{sc}}\}$ can be obtained by solving Eqs. (25) and (27)–(29).

Choosing $1 \leqslant a,b < a+b \leqslant n$ in Eq. (29), we have

$$\begin{aligned} 0 &\leqslant \Delta_3(a,b,n-a-b) \\ &= (h_{n-a-b}^{\mathrm{sc}} - h_{a+b}^{\mathrm{sc}}) - (h_{n-a}^{\mathrm{sc}} - h_a^{\mathrm{sc}}) - (h_{n-b}^{\mathrm{sc}} - h_b^{\mathrm{sc}}) \\ &= -\Delta_3(n-a,n-b,a+b) \leqslant 0 \end{aligned}$$

which implies the following *reflection condition* on $\{h_a^{\mathrm{sc}}\}$:

$$h_a^{\mathrm{sc}} - h_{n-a}^{\mathrm{sc}} = a(h_1^{\mathrm{sc}} - h_{n-1}^{\mathrm{sc}}) = 0. \tag{30}$$

From Eq. (30) we see that partially solving conditions (29) reduces the number of independent variables characterizing a pattern of zeros from $n-1$ in $\{S_2,\ldots,S_n\}$ to $[\frac{n}{2}]$ in $\{h_1^{\mathrm{sc}},\ldots,h_{[n/2]}^{\mathrm{sc}}\}$. However, being a sequence of fractions rather than integers, $\{h_a^{\mathrm{sc}}\}$ labeling scheme imposes some difficulty in numerically solving conditions (25) and (27)–(29). In Appendixes A 1 and A 2 we will further use consistent conditions (29) to introduce two schemes labeling the pattern of zeros with a sequence of non-negative integers or half-integers. They turn out to be quite efficient for numerical studies since consistent conditions (25) and (27)–(29) can be reduced to a much smaller set after introducing a new labeling scheme $\{M_k;p;m\}$ as in Appendix A 2. In particular, this $\{M_k;p;m\}$ labeling scheme is the same one as adopted in the literature of parafermion vertex algebra.[35]

III. CONSTRUCTING FQH WAVE FUNCTIONS FROM Z_N VERTEX ALGEBRAS

If we use $\{n;m;h_a^{\mathrm{sc}}\}$ to characterize n-cluster symmetric polynomial $\Phi(\{z_i\})$, conditions (27)–(29) are required by the single-valueness of the symmetric polynomial. Or more precisely, Eqs. (27)–(29) come from a simple requirement that the zeros in $\Phi(\{z_i\})$ all have integer orders. However, conditions (27)–(29) are incomplete in the sense that some patterns of zeros $\{n;m;h_a^{\mathrm{sc}}\}$ can satisfy those conditions but still do not correspond to any valid polynomial.

A. FQH wave function as a correlation function in Z_n vertex algebra

To find more consistent conditions, in the rest of this paper, we will introduce a new requirement for the symmetric polynomial. We require that the symmetric polynomial can be expressed as a correlation function in a vertex algebra. More specifically, we have[21–23]

$$\Phi(\{z_i\}) = \lim_{z_\infty\to\infty} z_\infty^{2h_N}\left\langle V(z_\infty)\prod_{i=1}^{N} V_e(z_i)\right\rangle, \tag{31}$$

where $V_e(z)$ is an electron operator and $V(\infty)$ represents a positive background to guarantee the charge neutral condition. This new requirement, or more precisely, the associativity of the vertex algebra, leads to new conditions on h_a^{sc}.

The electron operator has the following form:

$$V_e(z) = \psi(z) :e^{i\phi(z)/\sqrt{\nu}}:, \tag{32}$$

where: $e^{i\phi(z)/\sqrt{\nu}}$: (:: stands for normal ordering, which is implicitly understood hereafter) is a vertex operator in a Gaussian model. It has a scaling dimension of $\frac{1}{2\nu}$ and the following operator product expansion (OPE) (Ref. 36):

$$e^{ia\phi(z)}e^{ib\phi(w)} = (z-w)^{ab}e^{i(a+b)\phi(w)} + O((z-w)^{ab+1}). \tag{33}$$

The operator ψ is a primary field of Virasoro algebra obeys an quasi-Abelian fusion rule,

$$\psi_a\psi_b \sim \psi_{a+b} + \cdots, \quad \psi_a \equiv (\psi)^a, \tag{34}$$

where $\cdots$ represent other primary fields of Virasoro algebra whose scaling dimensions are higher than that of ψ_{a+b} by some integer values. We believe that the integral difference of the scaling dimensions is necessary to produce a single-valued correlation function [see Eq. (31)].

Let $\tilde{h}_a^{\mathrm{sc}}$ be the scaling dimension of the simple current ψ_a. Therefore the a-cluster operator

$$V_a \equiv (V_e)^a = \psi_a(z)e^{ia\phi(z)/\sqrt{\nu}} \tag{35}$$

has a scaling dimension of

$$h_a = \tilde{h}_a^{\mathrm{sc}} + \frac{a^2}{2\nu}. \tag{36}$$

The vertex algebra is defined through the following OPE of the a-cluster operators:

$$V_a(z)V_b(w) = C_{a,b}^{V}\frac{V_{a,b}(w)}{(z-w)^{h_a+h_b-h_{a+b}}} + O((z-w)^{h_{a+b}-h_a-h_b+1}), \tag{37}$$

where $C_{a,b}^{V}$ are the structure constants. However, the above OPE is not quite enough. To fully define the vertex algebra, we also need to define the relation between $V_a(z)V_b(w)$ and $V_b(w)V_a(z)$.

The correlation functions is calculated through the expectation value of radial-ordered operator product.[26,36,37] The radial-ordered operator product is defined through

$$(z-w)^{\alpha_{V_aV_b}}R[V_a(z)V_b(w)] = \begin{cases} (z-w)^{\alpha_{V_aV_b}}V_a(z)V_b(w), & |z|>|w| \\ \mu_{ab}(w-z)^{\alpha_{V_aV_b}}V_b(w)V_a(z), & |z|<|w| \end{cases} \quad (38)$$

where

$$\alpha_{V_aV_b} = h_a + h_b - h_{a+b}. \quad (39)$$

Note that the extra complex factor μ_{ab} is introduced in the above definition of radial order. In the case of standard conformal algebras, where $\alpha_{V_aV_b} \in \mathbb{Z}$, we choose $\mu_{ab}=-e^{i\pi\alpha_{V_aV_b}}$ if both V_a and V_b are fermionic and $\mu_{ab}=e^{i\pi\alpha_{V_aV_b}}$ if at least one of them is bosonic. But in general, the commutation factor can be different from ± 1 and can be chosen more arbitrarily.

To gain an intuitive understanding of the above definition of radial order, let us consider the Gaussian model and choose $V_a=e^{ia\phi}$ and $V_b=e^{ib\phi}$. The scaling dimensions of V_a and V_b are $h_a=\frac{a^2}{2}$ and $h_b=\frac{b^2}{2}$. $\alpha_{V_aV_b}=h_a+h_b-h_{a+b}$. We see that $\alpha_{V_aV_b} \in \mathbb{Z}$ if $a,b \in \mathbb{Z}$ and such a Gaussian model is an example of standard conformal algebras. If both a and b are odd, then h_a and h_b are half-integers and V_a and V_b are fermionic operators. In this case $\alpha_{V_aV_b}=-ab=$odd. So under the standard choice $\mu_{ab}=-e^{i\pi\alpha_{V_aV_b}}$, we have $\mu_{ab}=1$. If one of a and b is even, $\alpha_{V_aV_b}=-ab=$even and one of V_a and V_b is bosonic operators. Under the standard choice $\mu_{ab}=e^{i\alpha_{V_aV_b}}$, we have again $\mu_{ab}=1$. Even when a and b are not integers, in the Gaussian model, the radial order of $V_a=e^{ia\phi}$ and $V_b=e^{ib\phi}$ is still defined with a choice $\mu_{ab}=1$. This is a part of the definition of the Gaussian model. In this paper, we will choose a more general definition of radial order where μ_{ab} are assumed to be generic complex phases $|\mu_{ab}|=1$.

The vertex algebra generated by ψ have a form

$$\psi_a(z)\psi_b(w) = C_{a,b}\frac{\psi_{a+b}(w)}{(z-w)^{\tilde{h}_a^{\mathrm{sc}}+\tilde{h}_b^{\mathrm{sc}}-\tilde{h}_{a+b}^{\mathrm{sc}}}} + O((z-w)^{\tilde{h}_{a+b}^{\mathrm{sc}}-\tilde{h}_a^{\mathrm{sc}}-\tilde{h}_b^{\mathrm{sc}}+1}), \quad (40)$$

where

$$C_{a,b} \neq 0. \quad (41)$$

When combined with the U(1) Gaussian model, the above vertex algebra can produce the wave function for a FQH state [see Eq. (31)].

We will also limit ourselves to the vertex algebra that satisfies the n-cluster condition,

$$\psi_n = 1, \quad (42)$$

where 1 stands for the identity operator defined in Appendix B. Those vertex algebras are in some sense "finite" and correspond to rational conformal field theory. We will call such vertex algebra Z_n vertex algebra. We see that in general, a FQH state can be described by the direct product of a U(1) Gaussian model and a Z_n vertex algebra. Some exmaples of Z_n vertex algebra are studied in Refs. 38 and 39.

Note that the Z_n vertex algebras are different from the Z_n simple-current vertex algebras that will be defined in Sec. V. The Z_n simple-current vertex algebras are special cases of the Z_n vertex algebras. In this and the next sections, we will consider Z_n vertex algebras. We will further limit ourselves to Z_n simple-current vertex algebras in Sec. V and later.

As a result

$$\tilde{h}_a^{\mathrm{sc}} = \tilde{h}_{a+n}^{\mathrm{sc}}, \quad \tilde{h}_n^{\mathrm{sc}} = 0,$$

$$\mu_{ab} = \mu_{a+n,b} = \mu_{a,b+n}, \quad \mu_{n,a} = \mu_{a,n} = 1,$$

$$C_{a,b} = C_{a+n,b} = C_{a,b+n}, \quad C_{n,a} = C_{a,n} = 1,$$

$$C_{a,b} = \mu_{a,b}C_{b,a}. \quad (43)$$

By choosing proper normalizations for the operators ψ_a, we can have

$$C_{a,-a} = \begin{cases} 1, & a \bmod n \leqslant n/2 \\ \mu_{a,-a}, & a \bmod n > n/2, \end{cases}$$

$$C_{a,b} = 1 \quad \text{if } a \text{ or } b = 0 \bmod n. \quad (44)$$

To summarize, we see that the Z_n vertex algebras (whose correlation functions give rise to electron wave functions) are characterized by the following set of data $\{n;m;\tilde{h}_a^{\mathrm{sc}};C_{a,b},\ldots|a,b=1,\ldots,n\}$, where $m=n/\nu$. Here $\cdots$ represent other structure constants in the subleading terms. The commutation factors μ_{ab} are not included in the above data because they can be expressed in terms of $\tilde{h}_a^{\mathrm{sc}}$ and are not independent [see Eq. (E8)]. Since the Z_n vertex algebra encodes the many-body wave function of electrons, we can say that the data $\{n;m;\tilde{h}_a^{\mathrm{sc}};C_{a,b},\ldots|a,b=1,\ldots,n\}$ also characterize the electron wave function. We can study all the properties of electron wave functions by studying the data $\{n;m;\tilde{h}_a^{\mathrm{sc}};C_{a,b},\ldots|a,b=1,\ldots,n\}$. In the pattern-of-zero approach, we use data $\{n;m;h_a^{\mathrm{sc}}\}$ to characterize the wave functions. We will see that the $\{n;m;\tilde{h}_a^{\mathrm{sc}};C_{a,b},\ldots|a,b=1,\ldots,n\}$ characterization is more complete, which allows us to obtain some new results.

B. Relation between $\tilde{h}_a^{\mathrm{sc}}$ and h_a^{sc}

What is the relation between the two characterizations: $\{n;m;h_a^{\mathrm{sc}}|a=1,\ldots,n\}$ and $\{n;m;\tilde{h}_a^{\mathrm{sc}};C_{ab}|a,b=1,\ldots,n\}$? The single valueness of the correlation function $\Phi(\{z_i\})$ requires that the zeros in $\Phi(\{z_i\})$ all have integer orders. In this section, we derive conditions on the scaling dimension $\tilde{h}_a^{\mathrm{sc}}$, just from this integral-zero condition. This allows us to find a simple relation between $\{n;m;h_a^{\mathrm{sc}}|a=1,\ldots,n\}$ and $\{n;m;\tilde{h}_a^{\mathrm{sc}};C_{ab}|a,b=1,\ldots,n\}$.

From the definition of D_{ab} and the OPE (37), we see that

$$\begin{aligned} D_{a,b} &\equiv S_{a+b} - S_a - S_b \\ &= h_{a+b} - h_a - h_b \\ &= \tilde{h}_{a+b}^{\mathrm{sc}} - \tilde{h}_a^{\mathrm{sc}} - \tilde{h}_b^{\mathrm{sc}} + \frac{ab}{\nu} \\ &= D_{b,a}. \end{aligned} \quad (45)$$

We see that $D_{1,n}=\frac{n}{\nu}$. So $\frac{n}{\nu}$ is an positive integer which is called m.

From Eq. (45), we can show that[11,12]

$$S_a=\sum_{i=1}^{a-1} D_{i,1}=h_a-ah_1=\widetilde{h}_a^{\rm sc}-a\widetilde{h}_1^{\rm sc}+\frac{a(a-1)}{2\nu} \tag{46}$$

and

$$\widetilde{h}_a^{\rm sc}=S_a-\frac{aS_n}{n}+\frac{am}{2}-\frac{a^2m}{2n}. \tag{47}$$

Therefore, the $h_a^{\rm sc}$ introduced before is nothing but the scaling dimensions $\widetilde{h}_a^{\rm sc}$ of the simple currents ψ_a [see Eq. (24)]. In the following, we will use $h_a^{\rm sc}$ to describe the scaling dimensions of ψ_a. Thus the data $\{n;m;\widetilde{h}_a^{\rm sc};C_{a,b}|a,b=1,\ldots,n\}$ can be rewritten as $\{n;m;h_a^{\rm sc};C_{a,b}|a,b=1,\ldots,n\}$. Those $h_a^{\rm sc}$ satisfy Eqs. (27)–(29).

As emphasized in Ref. 11 and 12, the conditions (27)–(29), although necessary, are not sufficient. In the following, we will try to find more conditions from the vertex algebra.

C. Conditions on $h_a^{\rm sc}$ and $C_{a,b}$ from the associativity of vertex algebra

The multipoint correlation of a Z_n vertex algebra can be obtained by fusing operators together, thus reducing the original problem to calculating a correlation of fewer points.[35] It is the associativity of this vertex algebra that guarantees any different ways of fusing operators would yield the same correlation in the end[26] so that the electron wave function would be single valued. The associativity of a Z_n vertex algebra requires $h_a^{\rm sc}$ and $C_{a,b}$ to satisfy many consistent conditions. Those are the extra consistent conditions we are looking for. The consistent conditions come from two sources. The first source is the consistent conditions on the commutation factors $\mu_{a,b}$ as discussed in Appendix B. When applied to our vertex algebra (40), we find that some consistent conditions on $\mu_{a,b}$ allow us to express $\mu_{a,b}$ in terms of $h_a^{\rm sc}$. Then other consistent conditions on $\mu_{a,b}$ will become consistent conditions on $h_a^{\rm sc}$ (see Appendix E 1). The second source is GJI for the vertex algebra (40) as discussed in Appendix E 2. We like to stress that the discussions so far are very general. The consistent conditions that we have obtained for generic Z_n vertex algebra are necessary conditions for any FQH states.

A detailed derivation of those conditions on $h_a^{\rm sc}$ and $C_{a,b}$ is given in Appendix E. Here we just summarize the new and old conditions in a compact form. The consistent conditions can be divided in two classes. The first type of consistent conditions act only on the pattern of zeros $\{n;m;h_a^{\rm sc}\}$ [see Eqs. (27)–(29), (E9), (E10), (E12), (E14), and (E31)],

$$nh_{2a}^{\rm sc}-2nah_1^{\rm sc}+a(2a-1)m=0 \bmod 2n,$$

$$m>0,\quad mn=\text{even},$$

$$h_{a+b}^{\rm sc}-h_a^{\rm sc}-h_b^{\rm sc}+\frac{abm}{n}\in\mathbb{N},$$

$$h_{a+b+c}^{\rm sc}-h_{a+b}^{\rm sc}-h_{b+c}^{\rm sc}-h_{a+c}^{\rm sc}+h_a^{\rm sc}+h_b^{\rm sc}+h_c^{\rm sc}\in\mathbb{N},$$

$$n\alpha_{1,1}=\text{even},$$

$$a^2\alpha_{1,1}-\alpha_{a,a}=\text{even},\quad \forall\, a=1,2,\ldots,n-1,$$

$$\Delta_3\left(\frac{n}{2},\frac{n}{2},\frac{n}{2}\right)=4h_{n/2}^{\rm sc}\neq 1\quad \text{if } n=\text{even}, \tag{48}$$

where $h_a^{\rm sc}=h_{a+n}^{\rm sc}$ and $\alpha_{a,b}=h_a^{\rm sc}+h_b^{\rm sc}-h_{a+b}^{\rm sc}$.

The second type of consistent conditions act on the structure constants [see Eqs. (E21), (E22), (E27), and (E28)]. For any a, b, and c,

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}=\mu_{a,b}C_{a,c}C_{b,a+c}\quad \text{if } \Delta_3(a,b,c)=0,$$

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}+\mu_{a,b}C_{a,c}C_{b,a+c}\quad \text{if } \Delta_3(a,b,c)=1, \tag{49}$$

where $\mu_{a,b}$ is a function of the pattern of zeros $\{h_a^{\rm sc}\}$,

$$\mu_{ij}=(-1)^{ij\alpha_{1,1}-\alpha_{i,j}}=\pm 1.$$

For any $a\neq n/2$

$$C_{a,-a}=C_{a,a}C_{2a,-a}=1\quad \text{if } \Delta_3(a,a,-a)=0,$$

$$2C_{a,-a}=C_{a,a}C_{2a,-a}\quad \text{if } \Delta_3(a,a,-a)=1. \tag{50}$$

Here $C_{a,b}$ satisfies the normalization condition (44). There may be additional conditions when $\Delta_3(a,b,c)\neq 0,1$. But we do not know how to derive those conditions systematically at this time.

IV. EXAMPLES OF GENERIC FQH STATES DESCRIBED BY THE Z_N VERTEX ALGEBRA

To obtain the examples of generic FQH states, we have numerical solved conditions (48). [We do not require Eq. (E16) to be satisfied in order to include some valid interesting solutions, like Gaffnian which violates Eq. (E16).] In this section, we list some of those solutions in terms of $\{n;m;h_a^{\rm sc}|a=1,\ldots,n-1\}$. First we note that, for two n-cluster symmetric polynomial Φ_1 and Φ_2 described by $\{n;m_1;h_{1,a}^{\rm sc}\}$ and $\{n;m_2;h_{2,a}^{\rm sc}\}$, the product $\Phi=\Phi_1\Phi_2$ is also an n-cluster symmetric polynomial. Φ is described by the pattern of zeros

$$\{n;m;h_a^{\rm sc}\}=\{n;m_1+m_2;h_{1,a}^{\rm sc}+h_{2,a}^{\rm sc}\}. \tag{51}$$

Most of the solutions can be decomposed according to Eq. (51). We will call the solutions that cannot be decomposed primitive solutions. We will only list those primitive solutions. We only searched solutions with a filling fraction $\nu\geqslant 1/4$. We can see that most solutions shown also satisfy condition (E16), which means they obey OPE (68) and correspond to special Z_n vertex algebras. However, some solutions such as a four-cluster state called Gaffnian, explicitly violates condition (E16) and their OPE's take the more general form (E18) and (E19). They are described by generic Z_n vertex algebras.

A. $n=1$ case

There is only one $n=1$ primitive solution,

$$n=1: \quad c=0,$$

$$\{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\{2;\},$$

$$\{p;M_1\cdots M_{n-1}\}=\{0;\},$$

$$\{n_0\cdots n_{m-1}\}=\{1\ 0\}. \tag{52}$$

It is $\nu=1/2$ Laughlin state. Note that $h_a^{\text{sc}}=0$, indicating that the simple-current part of vertex algebra is trivial and has a zero central charge $c=0$. The vertex algebra contains only the U(1) Gaussian part.

B. $n=2$ case

We note that the $n=1$ primitive solution also appears as a $n=2$ primitive solution. We find only one new $n=2$ primitive solution,

$$n=2: \quad c=1/2\ (Z_2 \text{ parafermion state}),$$

$$\{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\left\{2;\frac{1}{2}\right\},$$

$$\{p;M_1\cdots M_{n-1}\}=\{1;0\},$$

$$\{n_0\cdots n_{m-1}\}=\{2\ 0\}. \tag{53}$$

It is the $\nu=1$ Pfaffian state Φ_{Z_2}. The simple current part of the vertex algebra is a Z_2 parafermion CFT.

If we only use conditions (27)–(29) obtained in Refs. 11 and 12, then

$$n=2,$$

$$\{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\left\{2;\frac{1}{4}\right\},$$

$$\{p;M_1\cdots M_{n-1}\}=\left\{\frac{1}{2};0\right\},$$

$$\{n_0\cdots n_{m-1}\}=\{1\ 0\}. \tag{54}$$

will be a solution. Such a solution does not correspond to any symmetric polynomial, indicating that conditions (27)–(29) are incomplete. An extra condition (E12) from commutation factors remove such an incorrect solution.

C. $n=3$ case

Apart from the $n=1$ primitive solution, we find only one new $n=3$ primitive solution,

$$n=3: \quad c=4/5\ (Z_3 \text{ parafermion state}),$$

$$\{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\left\{2;\frac{2}{3}\ \frac{2}{3}\right\},$$

$$\{p;M_1\cdots M_{n-1}\}=\{2;0\ 0\},$$

$$\{n_0\cdots n_{m-1}\}=\{3\ 0\}. \tag{55}$$

It is the Z_3 parafermion state Φ_{Z_3}.

D. $n=4$ case

Apart from the $n=1$ primitive solutions, we find only two new $n=4$ primitive solutions using conditions (27)–(29), (E10), (E12), and (E14),

$$n=4: \quad c=1\ (Z_4 \text{ parafermion state}),$$

$$\{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\left\{2;\frac{3}{4}\ 1\ \frac{3}{4}\right\},$$

$$\{p;M_1\cdots M_{n-1}\}=\{2;0\ 0\ 0\},$$

$$\{n_0\cdots n_{m-1}\}=\{4\ 0\}. \tag{56}$$

which is the Z_4 parafermion state Φ_{Z_4}, and

$$n=4:$$

$$\{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\left\{2;\frac{1}{4}\ 0\ \frac{1}{4}\right\},$$

$$\{p;M_1\cdots M_{n-1}\}=\left\{1;\frac{1}{2}\ 1\ \frac{1}{2}\right\},$$

$$\{n_0\cdots n_{m-1}\}=\{1\ 0\ 1\ 0\}. \tag{57}$$

We like to point out that a nonprimitive solution $\{m;h_1^{\text{sc}},\ldots,h_{n-1}^{\text{sc}}\}=2\times\{2;\frac{1}{4},0,\frac{1}{4}\}=\{4;\frac{1}{2},0,\frac{1}{2}\}$ is the Z_2 parafermion state (the Pfaffian state). Consistent conditions from a study of useful GJI's show that it has central charge $c=1/2$ (the same as Z_2 Pfaffian state) and $\mu_{2,a}=1$, $\partial\psi_2=0$, indicating that $\psi_2=1$ is the identity operator here. In other words, this Z_4 simple-current vertex algebra is generated by a Z_2 simple current.

Another nonprimitive solution $\{m;h_1^{\text{sc}},\ldots,h_{n-1}^{\text{sc}}\}=3\times\{2;\frac{1}{4},0,\frac{1}{4}\}=\{6;\frac{3}{4},0,\frac{3}{4}\}$ is the Gaffnian state.[34] Gaffnian vertex algebra is a Z_4 simple-current vertex algebra with $\mu_{1,3}=\mu_{1,2}=\mu_{2,3}=-1$ and $\partial\psi_2=0$. In comparison with Z_4 Pfaffian, this Z_4 Gaffnian vertex algebra cannot be generated by any Z_2 simple current. This example will be analyzed in detail in Sec. VII.

E. Including conditions (49) and (50)

In the above, we only considered conditions (48). Those patterns of zeros that satisfy Eq. (48) may not satisfy conditions (49) and (50), i.e., one may not be able to find $C_{a,b}$ that satisfy Eqs. (49) and (50). However, we do not know how to check conditions (49) and (50) systematically. We have to check them on a case by case basis.

For the Z_2 and Z_3 parafermion states, we find that Eqs. (49) and (50) reduce to trivial identities after using Eq. (44). So the nontrivial $C_{1,1}$ and $C_{2,2}$ for the Z_3 parafermion vertex algebra cannot be determined from Eqs. (49) and (50), which means that conditions (49) and (50) can be satisfied by any choices of $C_{a,b}$ that are consistent with Eq. (44).

For the state with pattern of zeros $\{n;m;h_a^{sc}\}=\{4;2;\frac{1}{4}\ 0\ \frac{1}{4}\}$, we find that by choosing $(a,b,c)=(1,2,3)$ and (1,3,3) in Eq. (49), we can obtain the following equations:

$$C_{1,2}C_{3,3} = C_{2,3}C_{1,1} = -1,$$

$$1 = C_{3,3}C_{1,2} - 1. \tag{58}$$

Clearly no $\{C_{a,b}\}$ can satisfy the above two equations. Thus the $n=4$ pattern of zeros $\{m;h_a^{sc}\}=\{2;\frac{1}{4}\ 0\ \frac{1}{4}\}$ do not correspond to any valid symmetric polynomial. It's interesting to note that the $n=4$ pattern of zeros $\{m;h_a^{sc}\}=2\times\{2;\frac{1}{4}\ 0\ \frac{1}{4}\}=\{4;\frac{1}{2}\ 0\ \frac{1}{2}\}$ correspond to the Z_2 parafermion state and the $n=4$ pattern of zeros $\{m;h_a^{sc}\}=3\times\{2;\frac{1}{4}\ 0\ \frac{1}{4}\}=\{6;\frac{3}{4}\ 0\ \frac{3}{4}\}$ correspond to the Gaffnian state, both being valid symmetric polynomials.

For the state with pattern of zeros $\{n;m;h_a^{sc}\}=\{4;4;1\ 1\ 1\}$, we find that by choosing $(a,b,c)=(1,1,1)$, (1,1,2), and (1,2,0) in Eq. (49), we can obtain the following equations:

$$C_{1,1}C_{2,1} = C_{1,1}C_{1,2} + C_{1,1}C_{1,2},$$

$$C_{1,1} = C_{1,2} = C_{1,2},$$

$$C_{1,2} = C_{1,2} = -C_{2,1}, \tag{59}$$

which can be reduced to $-C_{1,1}^2=2C_{1,1}^2$. We see that the only solution is $C_{1,1}=C_{1,2}=C_{2,1}=0$, which is not allowed by Eq. (41). Thus the $n=4$ pattern of zeros $\{m;h_a^{sc}\}=\{4;1\ 1\ 1\}$ do not correspond to any valid symmetric polynomial.

F. Summary

In Refs. 11 and 12, we have seen that conditions (27)–(29) are not enough since they allow the following pattern of zeros $\{n;m;h_a^{sc}\}=\{2;1;\frac{1}{4}\}$. Such a pattern of zeros does not correspond to any valid polynomial. Condition (48) obtained in this paper rule out the above invalid solution. So conditions (48) is more complete than conditions (27)–(29). However, condition (48) is still incomplete since they allow the invalid patterns of zeros such as $\{n;m;h_a^{sc}\}=\{4;2;\ \frac{1}{4}\ 0\ \frac{1}{4}\}$ and {4;4; 1 1 1}. Both of them can be ruled out by conditions (49) and (50).

Conditions (48)–(50) are the consistent conditions that we can find from some of GJI, based on the most general form of OPE (40). So those conditions are necessary but may not be sufficient.

The correspondence between the patterns of zeros $\{n;m;h_a^{sc}\}$ and FQH states is not one to one. There can be many polynomials that have the same pattern of zeros. This is not surprising since the pattern of zeros only fixes the highest-order zeros in electron wave functions (symmetric polynomials), while different patterns of lower-order zeros could lead to different polynomials in principle. In other words, the leading-order OPE (40) alone might not suffice to uniquely determine the correlation function of the vertex algebra. The examples studied in this section support such a belief. Explicit calculations for some examples suggest that the pattern of zeros *together with* the central charge c and simple current condition would uniquely determine the FQH state. This is a reason why we introduce Z_n simple-current vertex algebra in the next section.

V. Z_n SIMPLE-CURRENT VERTEX ALGEBRA

In the last section, we discuss "legal" patterns of zeros that satisfy the consistent conditions (48)–(50) and describe existing FQH states. If we believe that a legal pattern of zeros $\{n;m;h_a^{sc}\}$, or more precisely the data $\{n;m;h_a^{sc};c\}$, can completely describe a FQH state, then we should be able to calculate all the topological properties of the FQH states. But so far, from the pattern of zeros $\{n;m;h_a^{sc}\}$, we can only calculate the number of different quasiparticle types, quasiparticle charges, and the fusion algebra between the quasiparticles.[12,13] Even with the more complete data $\{n;m;h_a^{sc};c\}$, we still do not know, at this time, how to calculate the quasiparticle statistics and scaling dimensions.

One idea to calculate more topological properties from the data $\{n;m;h_a^{sc};c\}$ is to use the data to define and construct the corresponding Z_n vertex algebra, and then use the Z_n vertex algebra to calculate the quasiparticle scaling dimensions and the central charge c. However, so far we do not know how to use the data $\{n;m;h_a^{sc};c\}$ to completely construct a Z_n vertex algebra in a systematic manner.

Starting from this section, we will concentrate on a subset of legal patterns of zeros that correspond to a subset of Z_n vertex algebra. Such a subset is called Z_n simple-current vertex algebras. The FQH states described by those Z_n simple-current vertex algebras are called Z_n simple-current states. We will show that in many cases the quasiparticle scaling dimensions and the central charge c can be calculated from the data $\{n;m;h_a^{sc};c\}$ for those Z_n simple-current states.

A. OPE's of Z_n simple-current vertex algebra

The Z_n simple-current vertex algebra is defined through an Abelian fusion rule with cyclic Z_n symmetry for primary fields $\{\psi_a\}$ of Virasoro algebra,[35,40]

$$\psi_a\psi_b \sim \psi_{a+b}, \quad \psi_a \equiv (\psi)^a. \tag{60}$$

Compared to Eq. (34), here we require that ψ_a and ψ_b fuse into a single primary field of Virasoro algebra ψ_{a+b}. Such operators are called simple currents. The Z_n simple-current vertex algebra is defined by the following OPE of ψ_a,[35,40]

$$\psi_a(z)\psi_b(w) = C_{a,b}\frac{\psi_{a+b}(w)}{(z-w)^{\alpha_{a,b}}} + O((z-w)^{1-\alpha_{a,b}}), \tag{61}$$

$$\psi_a(z)\psi_{-a}(w) = \frac{1 + \frac{2h_a^{sc}}{c}(z-w)^2 T(w)}{(z-w)^{2h_a^{sc}}} + O((z-w)^{3-2h_a^{sc}}), \tag{62}$$

where we define

$$\alpha_{a,b} \equiv h_a^{sc} + h_b^{sc} - h_{a+b}^{sc}, \tag{63}$$

$\psi_{-a} \equiv \psi_{n-a}$ and $\psi_a = \psi_{n+a}$ is understood due to the Z_n symmetry. In the context the subscript a of Z_n simple currents is always defined as $a \bmod n$.

We like to point out here that the form of the OPE (62) is a special property of the Z_n simple-current vertex algebra. For a more general Z_n vertex algebra that describes a generic FQH state, the correspond OPE has a more general form,

$$\psi_a(z)\psi_{-a}(w) = \frac{1 + \frac{2h_a^{sc}}{c}(z-w)^2 T(w)}{(z-w)^{2h_a^{sc}}} + \frac{T'_a}{(z-w)^{2h_a^{sc}-2}} + O((z-w)^{3-2h_a^{sc}}), \tag{64}$$

where T'_a are dimension-2 primary fields of Virasoro algebra ($\{T'_a, a=1,\ldots,[\frac{n}{2}]\}$ may not be linearly independent though). Also, for the Z_n simple-current vertex algebra, the subleading terms in Eq. (61) are also determined. For more details, see Appendix F.

$\{C_{a,b}\}$ are the structure constants of this vertex algebra. We also have conformal symmetry

$$T(z)\psi_a(w) = \frac{h_a^{sc}}{(z-w)^2}\psi_a(w) + \frac{1}{z-w}\partial\psi_a(w) + O(1), \tag{65}$$

and Virasoro algebra

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + O(1) \tag{66}$$

where $T(z)$ represents the energy-momentum tensor, which has a scaling dimension of 2. c stands for the central charge as usual, which is also a structure constant.

Using the notation of generalized vertex algebra[26] (see Appendix B), we have

$$[\psi_i\psi_j]_{\alpha_{i,j}} = C_{i,j}\psi_{i+j}, \quad i+j \neq 0 \bmod n, \tag{67}$$

$$[\psi_i\psi_{-i}]_{\alpha_{i,-i}} = 1, \quad [\psi_i\psi_{-i}]_{\alpha_{i,-i}-1} = 0,$$

$$[\psi_i\psi_{-i}]_{\alpha_{i,-i}-2} = \frac{2h_i^{sc}}{c}T, \tag{68}$$

$$[T\psi_i]_2 = h_i^{sc}\psi_i = [\psi_i T]_2, \quad [T\psi_i]_1 = \partial\psi_i,$$

$$[\psi_i T]_1 = (h_i^{sc}-1)\partial\psi_i, \tag{69}$$

$$[TT]_4 = \frac{c}{2}, \quad [TT]_3 = 0, \quad [TT]_2 = 2T, \quad [TT]_1 = \partial T, \tag{70}$$

with $\alpha_{T,\psi_i}=2$ and $\alpha_{T,T}=4$. We call it a *special Z_n simple-current vertex algebra* if it satisfies OPE's (67)–(70). For example, the Z_n parafermion states[17] correspond to a series of special Z_n simple-current vertex algebras.

The commutation factor μ_{AB} equals unity if either A or B is the energy-momentum tensor T: $\mu_{T,\psi_i}=\mu_{\psi_i,T}=\mu_{T,T}=1$. Similarly we have $\mu_{A,1}=\mu_{1,A}=1$ for the identity operator 1 and any operator A. However, $\mu_{i,j}\equiv\mu_{\psi_i,\psi_j}$ given in Eq. (E8) can be ± 1 in general. In deriving OPE (68) we have assumed that $\mu_{i,-i}=1$, $\forall i$, which is not necessary. For example, the Z_4 Gaffnian does not satisfy $\mu_{i,-i}=1$, $\forall i$. So, we will adopt the more general OPE (E18) and (E19) instead of Eq. (68) to include examples like Gaffnian which do give a FQH wave function. OPE (68) is for a special Z_n simple-current vertex algebra that satisfies $\mu_{i,-i}=1$, $\forall i$. For a more general Z_n simple-current vertex algebra, they become

$$[\psi_i\psi_{-i}]_{\alpha_{i,-i}} = C_{i,-i}, \quad [\psi_i\psi_{-i}]_{\alpha_{i,-i}-1} = 0,$$

$$[\psi_i\psi_{-i}]_{\alpha_{i,-i}-2} = \frac{2C_{i,-i}h_i^{sc}}{c}T,$$

$$C_{i,-i} = \begin{cases} 1, & i \leqslant n/2 \bmod n \\ \mu_{i,-i}, & i > n/2 \bmod n \end{cases} \tag{71}$$

so that we always have $C_{a,b}=\mu_{a,b}C_{b,a}$ for any subscripts a and b in such an associative vertex algebra.

OPE's (67), (71), (69), (70), (116), and (117) define the *generalized Z_n simple-current vertex algebra*, or simply Z_n simple-current vertex algebra. The Gaffnian state corresponds to a generalized Z_4 simple-current vertex algebra with $\mu_{a,-a}\neq 1$. When $\mu_{a,-a}=1$, we have a special Z_n simple-current vertex algebra.

What kind of pattern of zeros $\{n;m;h_a^{sc}\}$, or more precisely what kind of data $\{n;m;h_a^{sc};c,C_{ab}\}$, can produce a Z_n simple-current vertex algebra? Since the Z_n simple-current vertex algebras are special cases of Z_n vertex algebras, the data $\{n;m;h_a^{sc};c,C_{ab}\}$ must satisfy conditions (48)–(50). However, the data $\{n;m;h_a^{sc};c,C_{ab}\}$ for Z_n simple-current vertex algebras should satisfy more conditions. Those conditions can be obtained from the GJI of Z_n simple-current vertex algebras. In Appendix E 2, we derived all those extra consistent conditions for a generic Z_n vertex algebra from the useful GJI's based on OPE (40). Now based on OPE's summarized in this section, we can similarly derive a set of extra consistent conditions for a Z_n simple-current vertex algebra. These conditions are summarized in Sec. V B. For those valid data that satisfy all the consistent conditions, the full properties of simple-current vertex algebra can be obtained. This in turn allows us to calculate the physical topological properties of the FQH states associated with those valid patterns of zeros.

We like to point out that many examples of Z_n simple-current vertex algebra have been studied in detail. They include the simplest Z_n simple-current vertex algebra—the Z_n parafermion algebra.[35,40,41] More general exmaples that have been studied are the higher generations of Z_n parafermion algebra[42–46] and graded parafermion algebra.[47–49] In those exapmles, the Z_n simple-current algebras are studied by embedding the algebras into some known CFT, such as coset models of Kac-Moody current algebras and/or Coulomb gas models. However, in this paper, we will not assume such kind of embeding. We will try to calculate the properties of Z_n simple-current vertex algebra directly from the data $\{n;m;h_a^{\text{sc}},c,\ldots\}$ without assuming any embedding.

B. Consistent conditions from useful GJI's

In Appendix E 2 a, we show how to obtain the consistent conditions on the data $\{n;m;h_a^{\text{sc}};c,C_{ab}\}$ characterizing a generic Z_n vertex algebra from a set of useful GJI's as described in Appendix D, requiring that OPE (E1) is obeyed. Here for a Z_n simple-current vertex algebra, requiring that OPE's (67), (71), (69), (70), (116), and (117) are obeyed, we can derive a larger set of consistent conditions on the data $\{n;m;h_a^{\text{sc}};c,C_{ab}\}$. For the examples studied in this paper, we find that C_{ab} can be uniquely determined from $\{n;m;h_a^{\text{sc}};c\}$ using those consistent conditions. Thus, for those states, C_{ab} are not independent and can be dropped.

Since $\Delta_3(A,B,C)=\Delta_3(A,C,B)=\Delta_3(C,A,B)$ for an associative vertex algebra, we can combine the consistent conditions obtained from GJI's of all possible six permutations of three operators (A,B,C) together. In this section we summarize the consistent conditions obtained from useful GJI's (just like in Appendix E 2 a) and list them in a compact manner. These extra consistent conditions, together with conditions (48) should form a complete set of consistent conditions, which allows us to obtain a valid pattern of zeros and construct the associated simple-current vertex algebra and FQH wave function.

1. $\{A,B,C\}=\{\psi_a,\psi_b,\psi_c\}$, $a+b,b+c,a+c\neq 0 \bmod n$

For $\Delta_3(a,b,c)=0$, we have the following consistent conditions:

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}=\mu_{a,b}C_{a,c}C_{b,a+c}. \tag{72}$$

Notice that the consistent conditions obtained from useful GJI's of (ψ_a,ψ_b,ψ_c) and of (ψ_b,ψ_a,ψ_c) only differ by a factor of $\mu_{a,b}$ since $C_{a,b}=\mu_{a,b}C_{b,a}$. Thus they are not independent conditions. Similarly it's easy to show that other permutations yield consistent conditions linearly dependent with the above condition using the fact that $\mu_{a,b}\mu_{a,c}=\mu_{a,b+c}$ here since $\Delta_3(a,b,c)=0$.

For $\Delta_3(a,b,c)=1$, we have the following consistent conditions:

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}+\mu_{a,b}C_{a,c}C_{b,a+c}. \tag{73}$$

For $\Delta_3(a,b,c)\geqslant 2$ there are no extra consistent conditions.

2. $\{A,B,C\}=\{\psi_a,\psi_b,\psi_{-b}\}$, $a+b\neq 0 \bmod n$

For $\Delta_3(a,b,-b)=0$ we have the following independent consistent conditions,

$$h_a^{\text{sc}}h_b^{\text{sc}}=\alpha_{a,\pm b}=0,\quad h_b^{\text{sc}}\partial\psi_a\equiv 0,$$

$$C_{a,b}C_{a+b,-b}=\mu_{a,b}C_{a,-b}C_{b,a-b}=C_{b,-b} \tag{74}$$

since we know Eq. (B7) and $\mu_{a,0}=1$.

For $\Delta_3(a,b,-b)=1$ the independent consistent conditions are

$$c=\frac{4h_a^{\text{sc}}h_b^{\text{sc}}}{\alpha_{a,b}(1-\alpha_{a,b})},$$

$$C_{a,b}C_{a+b.-b}=(1-\alpha_{a,b})C_{b,-b},$$

$$\mu_{a,b}C_{a,-b}C_{b,a-b}=-\alpha_{a,b}C_{b,-b}. \tag{75}$$

For $\Delta_3(a,b,-b)=2$ the independent consistent conditions are

$$\mu_{a,b}C_{a,-b}C_{b,a-b}=\left[\frac{\alpha_{a,b}(\alpha_{a,b}-1)}{2}+\frac{2h_a^{\text{sc}}h_b^{\text{sc}}}{c}\right]C_{b,-b},$$

$$C_{a,b}C_{a+b,-b}=\left[\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h_a^{\text{sc}}h_b^{\text{sc}}}{c}\right]C_{b,-b}. \tag{76}$$

For $\Delta_3(a,b,-b)=3$ the independent consistent conditions are

$$C_{a,b}C_{a+b,-b}-\mu_{a,b}C_{a,-b}C_{b,a-b}=\left[\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h_a^{\text{sc}}h_b^{\text{sc}}}{c}\right]C_{b,-b}. \tag{77}$$

For $\Delta_3(a,b,-b)\geqslant 4$ there are no extra consistent conditions from useful GJI's.

3. $\{A,B,C\}=\{\psi_a,\psi_a,\psi_{-a}\}$, $a\neq n/2 \bmod n$

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{\text{sc}}=0$ the consistent conditions are summarized as

$$h_a^{\text{sc}}=\alpha_{a,a}=0,\quad \partial\psi_a\equiv 0,$$

$$C_{a,a}C_{2a,-a}=C_{a,-a}=C_{-a,a}=\mu_{a,-a}=1. \tag{78}$$

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{\text{sc}}=1$ the corresponding consistent conditions are

$$\alpha_{a,a}=-1,\quad h_a^{\text{sc}}=1,\quad c=-2,\quad \mu_{a,-a}=-1,$$

$$C_{a,a}C_{2a,-a}=2C_{a,-a},\quad C_{a,-a}=-C_{-a,a}. \tag{79}$$

For $\Delta_3(a,a,-a)=2$ the independent consistent conditions are

$$c=\frac{2h_a^{\text{sc}}}{3-2h_a^{\text{sc}}},$$

$$C_{a,a}C_{2a,-a}=2h_a^{\text{sc}},\quad C_{a,-a}=C_{-a,a}=\mu_{a,-a}=1 \tag{80}$$

since we have $C_{2a,-a}=C_{-a,2a}$ here.

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{\text{sc}}=3$ the extra consistent conditions are

$$\mu_{a,-a}=-1,$$

$$c=\frac{-2(h_a^{\mathrm{sc}})^2}{(2h_a^{\mathrm{sc}}-3)(h_a^{\mathrm{sc}}-2)}, \quad C_{-a,a}=-C_{a,-a},$$

$$C_{a,a}C_{2a,-a}=4(h_a^{\mathrm{sc}}-1)C_{a,-a}. \tag{81}$$

For $\Delta_3(a,a,-a)=4$ the independent consistent conditions are

$$C_{a,-a}=C_{-a,a}=\mu_{a,-a}=1,$$

$$C_{a,a}C_{2a,-a}=h_a^{\mathrm{sc}}(2h_a^{\mathrm{sc}}-3)+\frac{2(h_a^{\mathrm{sc}})^2}{c}. \tag{82}$$

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{\mathrm{sc}}=5$ there is only 1 useful GJI and the consistent conditions is

$$\mu_{a,-a}=-1, \quad C_{a,-a}=-C_{-a,a},$$

$$C_{a,a}C_{2a,-a}=2\left[(h_a^{\mathrm{sc}}-2)(2h_a^{\mathrm{sc}}-3)+\frac{2(h_a^{\mathrm{sc}})^2}{c}\right]C_{a,-a}.$$

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{\mathrm{sc}}\geqslant 6$ there are no useful GJI's and no extra consistent conditions.

4. $\{A,B,C\}=\{\psi_{n/2},\psi_{n/2},\psi_{n/2}\}$, n=even

Just like shown in Appendix E 2 a, we require that

$$\Delta_3(n/2,n/2,n/2)\neq 1,3,5. \tag{83}$$

Otherwise the useful GJI's would yield a contradiction $\psi_{n/2}=0$.

For $\Delta_3(n/2,n/2,n/2)=0$ the extra consistent conditions are

$$h_{n/2}^{\mathrm{sc}}=0, \quad \partial\psi_{n/2}\equiv 0. \tag{84}$$

For $\Delta_3(n/2,n/2,n/2)=2$ the extra consistent conditions are

$$c=h_{n/2}^{\mathrm{sc}}=1/2. \tag{85}$$

For $\Delta_3(n/2,n/2,n/2)=4$ the extra consistent conditions are

$$c=h_{n/2}^{\mathrm{sc}}=1. \tag{86}$$

For $\Delta_3(n/2,n/2,n/2)=6$ there are no extra consistent conditions.

For $\Delta_3(n/2,n/2,n/2)=7$ the extra consistent conditions are

$$c=49, \quad h_{n/2}^{\mathrm{sc}}=7/4. \tag{87}$$

For $\Delta_3(n/2,n/2,n/2)\geqslant 8$ there are no extra consistent conditions.

VI. REPRESENTING QUASIPARTICLES IN Z_n SIMPLE-CURRENT VERTEX ALGEBRA

Since the Z_n simple-current vertex algebras completely determine the FQH states and their topological orders, we should be able to calculate all the topological properties from the vertex algebras. In this section, we will discuss how to represent quasiparticles and how to calculate quasiparticle properties from the vertex algebras.

A. Pattern of zeros for quasiparticles and its consistent conditions

First, let us review the pattern of zeros description for quasiparticles in FQH states.[12,13]

1. Definition and consistent conditions

The pattern of zeros for the ground-state wave function can be easily generalized to describe the wave functions with quasiparticle excitations. If a symmetric polynomial $\Phi(\{z_i\})$ has a quasiparticle at $z=0$, $\Phi(\{z_i\})$ will have a different pattern of zeros $\{S_{\gamma;a}\}$ as $z_1=\lambda\xi_1,\ldots,z_a=\lambda\xi_a$ approach 0,

$$\lim_{\lambda\to 0^+}\Phi(\{z_i\})=\lambda^{S_{\gamma;a}}P_\gamma(\xi_1,\ldots,\xi_a;z_{a+1},\ldots)+O(\lambda^{S_{\gamma;a}+1}). \tag{88}$$

Thus we can use the sequence of non-negative integers $\{S_{\gamma;a}\}$ to quantitatively characterize quasiparticles.

It was shown[12,13] that there are similar consistent conditions on the quasiparticle pattern of zeros $\{S_{\gamma;a}\}$.

First concave condition,

$$D_{\gamma+a,b}\equiv S_{\gamma;a+b}-S_{\gamma;a}-S_b\geqslant 0. \tag{89}$$

Second concave condition,

$$\Delta_3(\gamma+a;b,c)\equiv S_{\gamma;a+b+c}+S_{\gamma;a}+S_b+S_c-S_{\gamma;a+b}-S_{\gamma;a+c}-S_{b+c}\geqslant 0, \tag{90}$$

n-cluster condition

$$S_{\gamma;a+kn}=S_{\gamma;a}+k(S_{\gamma;n}+ma)+mn\frac{k(k-1)}{2}. \tag{91}$$

$\{S_{\gamma;a}\}$ is a quantitative way to label all types of the quasiparticles in the FQH state described by $\{S_a\}$. The question is that is $\{S_{\gamma;a}\}$ an one-to-one label of the quasiparticles? Can two different quasiparticles share the same pattern of zeros? The answer is yes and no. For certain FQH states (such as all the generalized and composite parafermion FQH states), $\{S_{\gamma;a}\}$ is an one-to-one label of all the quasiparticles. While for other FQH states, such as $Z_2|Z_2$ and $Z_3|Z_3$ in Sec. VII, $\{S_{\gamma;a}\}$ is not an one-to-one label and two different quasiparticles can have the same pattern of zeros.

If we assume $\{S_{\gamma;a}\}$ to be an one-to-one label of all the quasiparticles, then by solving the above consistent conditions, we can obtain the number of quasiparticle types, which happens to equal the ground state degeneracy of the FQH state on a torus. We can also calculate other physical properties of quasiparticles from $\{S_{\gamma;a}\}$. For example, the quasiparticle charge Q_γ can be obtained from the pattern of zeros as[12]

$$Q_\gamma=\frac{S_{\gamma;n}-S_n}{m}. \tag{92}$$

(The above formula is valid even when $\{S_{\gamma;a}\}$ is not a one-to-one label.)

2. Label quasiparticle pattern of zeros by $\{k^{sc}_{\gamma;a};Q_\gamma\}$

Another way to label the quasiparticle pattern of zeros can be obtained by introducing the $\{k^{sc}_{\gamma;a}\}$ vector (which is denoted by $\tilde{l}^{sc}_{\gamma;a}$ in Ref. 13),

$$k^{sc}_{\gamma;a} \equiv S_{\gamma;a} - S_{\gamma;a-1} + h^{sc}_1 - \frac{m(Q_\gamma + a - 1)}{n}. \qquad (93)$$

Conversely we have

$$S_{\gamma;a} = \sum_{i=1}^{a} k^{sc}_{\gamma;i} + a\left(\frac{mQ_\gamma}{n} - h^{sc}_1\right) + \frac{ma(a-1)}{2n}. \qquad (94)$$

The n-cluster condition (91) of $\{S_{\gamma;a}\}$ sequence results in the periodic property of $\{k^{sc}_{\gamma;a}\}$

$$k^{sc}_{\gamma;n+a} = k^{sc}_{\gamma;a}. \qquad (95)$$

Therefore we can use the set of data $\{k^{sc}_{\gamma;1},\ldots,k^{sc}_{\gamma;n};Q_\gamma\}$ to describe quasiparticles.

Let $a=n$ in Eq. (94) and use Eq. (92) we can see that

$$\sum_{i=1}^{n} k^{sc}_{\gamma;i} = 0. \qquad (96)$$

The two concave conditions (89) and (90) for this set of data now becomes

$$D_{\gamma+a,b} = \sum_{i=1}^{b} k^{sc}_{\gamma;a+i} - h^{sc}_b + b\frac{mQ_\gamma}{n} + \frac{mab}{n} \in \mathbb{N}, \qquad (97)$$

$$\Delta_3(\gamma+a,b,c) = \sum_{i=1}^{c} (k^{sc}_{\gamma;a+b+i} - k^{sc}_{\gamma;a+i}) + h^{sc}_b + h^{sc}_c - h^{sc}_{b+c} \in \mathbb{N}. \qquad (98)$$

A set of $\{k^{sc}_{\gamma;1},\ldots,k^{sc}_{\gamma;n};Q_\gamma\}$ satisfying the above two conditions and $S_{\gamma;a}\geqslant 0$ can generate a valid quasiparticle pattern of zeros, which corresponds to quasiparticle above a ground state with the pattern of zeros $\{h^{sc}_a\}$.

We note that $\gamma+1$ corresponds to a bound state between a γ quasiparticle and a hole (the absence of an electron). The $(\gamma+1)$ quasiparticle is labeled by

$$\{k^{sc}_{\gamma+1;1},\ldots,k^{sc}_{\gamma+1;n};Q_{\gamma+1}\} = \{k^{sc}_{\gamma;2},\ldots,k^{sc}_{\gamma;n},k^{sc}_{\gamma;1};Q_\gamma+1\}.$$

Since two quasiparticles that differ by an electron are regarded as equivalent, we can use the above equivalence relation to pick an equivalent label that has the minimal charge and satisfies $S_{\gamma;a}\geqslant 0$. For each equivalence class, there exists only one such label. In this paper, we will use such a label to label inequivalent quasiparticles.

B. Quasiparticle wave functions and quasiparticle operators

Just like the ground state wave function (31), the wave function with a quasiparticle can also be written as a R-ordered correlation function between electron operators and quasiparticle operators in the vertex algebra

$$\Phi_\gamma(w;\{z_i\}) = \lim_{z_\infty\to\infty} z_\infty^{2h_N}\left\langle V(z_\infty)\left[\prod_i V_e(z_i)\right]V_\gamma(w)\right\rangle, \quad (99)$$

where w is the location of the quasiparticle and $V_\gamma(w)$ is the quasiparticle operator. By definition, a quasiparticle operator can be any operator that is mutually local respect to the electron operators $V_e(z)$.

In our simple-current$\times$U(1) vertex algebra, the quasiparticle operator V_γ has the following form:

$$V_\gamma(z) = \sigma_\gamma(z) {:}e^{i\phi(z)Q_\gamma/\sqrt{\nu}}{:}, \qquad (100)$$

where σ_γ is a disorder operator[41] that generates a representation of the simple current part of the vertex algebra. When a electrons and one quasiparticle are fused together we have

$$V_{\gamma+a}(z) \propto V_a V_\gamma = \sigma_{\gamma+a}(z){:}e^{i\phi(z)(Q_\gamma+a)/\sqrt{\nu}}{:},$$

$$\sigma_{\gamma+a} \propto \psi_a\sigma. \qquad (101)$$

The OPE between the quasiparticle operator and the electron operator can be written as

$$V_e(z)V_{\gamma+a}(w) \propto (z-w)^{l_{\gamma;a+1}} V_{\gamma+a+1}(w) + \cdots. \qquad (102)$$

The mutual locality between the quasiparticle operator and the electron operator requires $l_{\gamma;a}$ to be integers. In order for the quasiparticle wave function $\Phi_\gamma(w;\{z_i\})$ to contain no poles, we also require that $l_{\gamma;a}\geqslant 0$.

In fact, the sequence $l_{\gamma;a}$, $a=1,2,\ldots$, provides a quantitative way to label the quasiparticles (and quasiparticle operators). We have introduced another quantitative label of the quasiparticles in terms of $S_{\gamma;a}$, $a=1,2,\ldots$ The two labeling schemes are related by[12,13]

$$S_{\gamma;a} = \sum_{i=1}^{a} l_{\gamma;i}, \quad l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}. \qquad (103)$$

We can also convert the orbital sequence $l_{\gamma;a}$ into an occupation sequence $n_{\gamma;l}$. If we view $l_{\gamma;a}$ as the index of the orbital occupied by the ath particle, then $n_{\gamma;l}$ is simply the number of particles occupying the lth orbital.

Let us denote the scaling dimension of disorder operators σ_γ as h^{sc}_γ. Can we calculate those scaling dimensions from the data $l_{\gamma;a}$ that characterize the quasiparticle? From the OPE of the quasiparticle operators, we find the following relations:

$$l_{\gamma;a+1} = h^{sc}_{\gamma+a+1} - h^{sc}_{\gamma+a} - h^{sc}_1 + \frac{m(Q_\gamma+a)}{n} \qquad (104)$$

and

$$S_{\gamma;a} = \sum_{i=1}^{a} l_{\gamma;i} = h^{sc}_{\gamma+a} - h^{sc}_\gamma + a\left(\frac{mQ_\gamma}{n} - h^{sc}_1\right) + \frac{ma(a-1)}{2n}, \qquad (105)$$

Making use of Eq. (93) we immediately obtain the relations between $\{k^{sc}_{\gamma;a}\}$ and $\{h^{sc}_{\gamma+b}\}$,

$$k^{sc}_{\gamma;a} = h^{sc}_{\gamma+a} - h^{sc}_{\gamma+a-1}, \qquad (106)$$

which implies that

$$h^{\text{sc}}_{\gamma+a} = h^{\text{sc}}_{\gamma} + \sum_{i=1}^{a} k^{\text{sc}}_{\gamma;i}. \tag{107}$$

Moreover, Eqs. (95) and Eq. (96) lead to the periodic condition on $h^{\text{sc}}_{\gamma+a}$,

$$h^{\text{sc}}_{\gamma+a+n} = h^{\text{sc}}_{\gamma+a}, \tag{108}$$

which is implied by the fusion rule $\psi_n\sigma=\sigma$ since $\psi_n=1$.

We know that we can use $\{k^{\text{sc}}_{\gamma;1},\ldots,k^{\text{sc}}_{\gamma;n};Q_\gamma\}$ that satisfies the two concave conditions (97) and (98) to describe (or label) a quasiparticle operator V_γ (or a quasiparticle γ). The above result (107) only allows us to determine the scaling dimensions $\{h^{\text{sc}}_{\gamma+a}\}$ of the associated disorder operators up to a constant. That is if we know the scaling dimension h^{sc}_{γ} of a disorder operator σ_γ, then the scaling dimensions of a family of disorder operators $\sigma_{\gamma+a}$ can be determined. However, the scaling dimension h^{sc}_{γ} cannot be determined from the considerations discussed here. Can we do a better job by fully using the structure of the vertex algebra? In Secs. VI D and VII we will show how to extract the scaling dimension $h^{\text{sc}}_{\gamma+a}$ from useful GJI's defined in Appendix D.

C. More complete characterization of quasiparticles

Through a study of Z_n vertex algebra, we have realized that the pattern-of-zero data $\{n;m;h^{\text{sc}}_a\}$ does not fully describe a symmetric polynomial (i.e., a quantum Hall wave function). We need to at least expand $\{n;m;h^{\text{sc}}_a\}$ to $\{n;m;h^{\text{sc}}_a;c\}$ to characterize a quantum Hall wave function. Similarly, the data $\{k^{\text{sc}}_{\gamma;a};Q_\gamma\}$ does not fully describe a quasiparticle either, i.e., some times, different quasiparticles can have the same pattern of zeros $\{k^{\text{sc}}_{\gamma;a};Q_\gamma\}$.

To see how to extend $\{k^{\text{sc}}_{\gamma;a};Q_\gamma\}$, we note that a generic OPE between $\sigma_{\gamma+b}$ and ψ_a has a form

$$\psi_a(z)\sigma_{\gamma+b}(w) = \frac{C_{a,\gamma+b}}{(z-w)^{\alpha_{a,\gamma+b}}}\sigma_{\gamma+a+b}(w) + \cdots, \tag{109}$$

where

$$\alpha_{a,\gamma+b} = h^{\text{sc}}_a + h^{\text{sc}}_{\gamma+b} - h^{\text{sc}}_{\gamma+a+b}. \tag{110}$$

We also need to introduce the commutation factor $\mu_{a,\gamma+b}$,

$$(z-w)^{\alpha_{a,\gamma+b}}\psi_a(z)\sigma_{\gamma+b}(w) = \mu_{a,\gamma+b}(w-z)^{\alpha_{a,\gamma+b}}\sigma_{\gamma+b}(w)\psi_a(z), \tag{111}$$

to describe the commutation relation between $\sigma_{\gamma+b}$ and ψ_a. We see that in vertex algebra, we need additional data, $C_{a,\gamma+b}$, $C_{\gamma+b,a}$, $\mu_{a,\gamma+b}$, and $\mu_{\gamma+b,a}$, to describe the quasiparticle γ. (In appendix C, we give a discussion about the relation between the quasiparticle commutation factor $\mu_{a,\gamma}$ and quasiparticle pattern of zeros $\{k^{\text{sc}}_{\gamma;a};Q_\gamma\}$.)

However, if we put the quasiparticle at $w=0$ [see Eq. (99)], then we do not need to use commutation factor $\mu_{a,\gamma}$ when we calculate the R-ordered correlation function (99). Thus, the electron wave function with a quasiparticle do not depend on the commutation factor $\mu_{a,\gamma}$. Similarly, the R-ordered correlation function only depend on $C_{a,\gamma+b}$. Therefore, we only need to add $C_{a,\gamma+b}$ to describe the quasiparticle γ more completely.

Therefore, within the simple-current vertex algebra, we can use the following more complete data:

$$\{k^{\text{sc}}_{\gamma;a};Q_\gamma;C_{a,\gamma+b}\} \tag{112}$$

to describe a quasiparticle. By considering the full structure of the vertex algebra (see Sec. VI D), we can obtain many self-consistent conditions on the data $\{k^{\text{sc}}_{\gamma;a};Q_\gamma;C_{a,\gamma+b}\}$. In particular, we can calculate the scaling dimension h^{sc}_γ of σ_γ from the data $\{k^{\text{sc}}_{\gamma;a};Q_\gamma;C_{a,\gamma+b}\}$.

Once we find the scaling dimension h^{sc}_γ of a disorder operator σ_γ, the scaling dimension h_γ of the associated quasiparticle operator V_γ can be determined from

$$h_\gamma = h^{\text{sc}}_\gamma + h^{\text{ga}}_\gamma = h^{\text{sc}}_\gamma + \frac{mQ^2_\gamma}{2n}. \tag{113}$$

where h^{ga}_γ is the scaling dimension of the U(1) part $e^{i\phi Q_\gamma/\sqrt{\nu}}$ of the quasiparticle operator. h_γ is the intrinsic spin of the quasiparticle which is closely related to the statistics of the quasiparticle. (Note that in 2+1D the intrinsic spin is not quantized as half-integer.)

D. Consistent conditions for quasiparticles from useful GJI's

1. Complete vertex algebra with quasiparticle operators

To find more consistent conditions on the quasiparticle data $\{k^{\text{sc}}_{\gamma;a};Q_\gamma;C_{a,\gamma+b}\}$, we need to write down the complete OPE between the disorder operators $\sigma_{\gamma+b}$ and the simple currents,

$$\psi_a(z)\sigma_{\gamma+b}(w) = C_{a,\gamma+b}\frac{\sigma_{\gamma+a+b}(w)}{(z-w)^{\alpha_{a,\gamma+b}}} + O((z-w)^{1-\alpha_{a,\gamma+b}}),$$

$$T(z)\sigma_{\gamma+a}(w) = \frac{h^{\text{sc}}_{\gamma+a}}{(z-w)^2}\sigma_{\gamma+a}(w) + \frac{1}{z-w}\partial\sigma_{\gamma+a}(w) + O(1), \tag{114}$$

where we define

$$\alpha_{a,\gamma+b} \equiv h^{\text{sc}}_a - (h^{\text{sc}}_{\gamma+a+b} - h^{\text{sc}}_{\gamma+b}) = h^{\text{sc}}_a - \sum_{i=1}^{a} k^{\text{sc}}_{\gamma;b+i}, \tag{115}$$

In other words we have

$$[\psi_a\sigma_{\gamma+b}]_{\alpha_{a,\gamma+b}} = C_{a,\gamma+b}\sigma_{\gamma+a+b},$$

$$[\sigma_{\gamma+b}\psi_a]_{\alpha_{a,\gamma+b}} = \mu_{\gamma+b,a}C_{a,\gamma+b}\sigma_{\gamma+a+b}, \tag{116}$$

$$[T\sigma_{\gamma+a}]_2 = h^{\text{sc}}_{\gamma+a}\sigma_{\gamma+a} = [\sigma_{\gamma+a}T]_2,$$

$$[T\sigma_{\gamma+a}]_1 = \partial\sigma_{\gamma+a}, \quad [\sigma_{\gamma+a}T]_1 = (h^{\text{sc}}_{\gamma+a}-1)\partial\sigma_{\gamma+a}, \tag{117}$$

with $\alpha_{T,\sigma_{\gamma+a}}=2$. We set $C_{a,\gamma}=1$ as the definition of disorder operators $\sigma_{\gamma+a}$, $a\neq 0 \bmod n$, which possess Z_n symmetry. Note that Eq. (117) can be used in GJI's to determine the scaling dimension $h^{\text{sc}}_{\gamma+a}$ of disorder operators, as will be shown in examples.

The consistent conditions on the quasiparticle data, $\{k^{\text{sc}}_{\gamma;a};Q_\gamma;C_{a,\gamma+b}\}$ or $\{h^{\text{sc}}_{\gamma+a};Q_\gamma;C_{a,\gamma+b}\}$, can also be obtained from useful GJI's with respect to the OPE's (116) and (117), just as we did in Appendix E 2 a for simple currents of a generic Z_n vertex algebra. In the following, we will list the obtained consistent conditions from GJI's.

2. Consistent conditions: $\{A,B,C\}=\{\psi_a,\psi_b,\sigma_{\gamma+c}\}$, $a+b \neq 0 \bmod n$

Apply the GJI to the quasiparticle algebra (116) and (117), we can obtain many new consistent conditions.

For $\Delta_3(a,b,\gamma+c)=0$ the independent consistent conditions are

$$\mu_{a,b}C_{a,\gamma+c}C_{b,\gamma+a+c}=C_{a,b}C_{a+b,\gamma+c}=C_{b,\gamma+c}C_{a,\gamma+b+c}. \tag{118}$$

For $\Delta_3(a,b,\gamma+c)=1$ the only independent consistent condition is

$$\mu_{a,b}C_{a,\gamma+c}C_{b,\gamma+a+c}=C_{a,b}C_{a+b,\gamma+c}-C_{b,\gamma+c}C_{a,\gamma+b+c}. \tag{119}$$

For $\Delta_3(a,b,\gamma+c)\geqslant 2$ there are no extra consistent conditions.

3. Consistent conditions: $\{A,B,C\}=\{\psi_a,\psi_{-a},\sigma_{\gamma+c}\}$

For $\Delta_3(a,-a,\gamma+b)=0$ the independent consistent conditions are

$$h_a^{\rm sc}\partial\sigma_{\gamma+b}\equiv 0,\quad \alpha_{\pm a,\gamma+b}=h_a^{\rm sc}h_{\gamma+b}^{\rm sc}=0,$$

$$C_{a,\gamma+b}C_{-a,\gamma+a+b}=C_{-a,\gamma+b}C_{a,\gamma+b-a}=C_{a,-a} \tag{120}$$

since $\mu_{\gamma+b,0}=1$.

For $\Delta_3(a,-a,\gamma+b)=1$ the independent consistent conditions are

$$\frac{h_a^{\rm sc}}{c}h_{\gamma+b}^{\rm sc}=\frac{\alpha_{a,\gamma+b}(1-\alpha_{a,\gamma+b})}{4},$$

$$C_{-a,\gamma+b}C_{a,\gamma+b-a}=C_{a,-a}\alpha_{a,\gamma+b},$$

$$C_{a,\gamma+b}C_{-a,\gamma+a+b}=C_{-a,a}\alpha_{-a,\gamma+b}. \tag{121}$$

Notice here the quasiparticle scaling dimension $h_\gamma^{\rm sc}$ is determined through useful GJI's.

For $\Delta_3(a,-a,\gamma+b)=2$ the independent consistent conditions are

$$C_{-a,\gamma+b}C_{a,\gamma+b-a}=\left[\frac{2h_a^{\rm sc}h_{\gamma+b}^{\rm sc}}{c}+\frac{\alpha_{a,\gamma+b}(\alpha_{a,\gamma+b}-1)}{2}\right]C_{a,-a},$$

$$C_{a,\gamma+b}C_{-a,\gamma+a+b}=\left[\frac{2h_a^{\rm sc}h_{\gamma+b}^{\rm sc}}{c}+\frac{(\alpha_{a,\gamma+b}-2)(\alpha_{a,\gamma+b}-1)}{2}\right]C_{-a,a}. \tag{122}$$

For $\Delta_3(a,-a,\gamma+b)=3$ the independent consistent condition is

$$\mu_{a,-a}C_{a,\gamma+b}C_{-a,\gamma+a+b}+C_{-a,\gamma+b}C_{a,\gamma+b-a}=\left[\frac{2h_a^{\rm sc}h_{\gamma+b}^{\rm sc}}{c}+\frac{(\alpha_{a,\gamma+b}-2)(\alpha_{a,\gamma+b}-1)}{2}\right]C_{a,-a}. \tag{123}$$

For $\Delta_3(a,-a,\gamma+b)\geqslant 4$ there are no extra consistent conditions from useful GJI's.

VII. EXAMPLES OF FQH STATES DESCRIBED BY Z_n SIMPLE-CURRENT VERTEX ALGEBRAS

In this section, we will examine some examples of FQH states that can be described by Z_n simple-current vertex algebra.

A. Pattern of zeros for Z_n simple-current vertex algebra

When we consider FQH states described by Z_n simple-current vertex algebra, the patterns of zero for those FQH states satisfy many additional conditions on top of conditions (48)–(50) for generic FQH states. In section V B, we list those additional consistent conditions obtained from GJI. Many conditions do not contain the structure constants C_{ab}, and those conditions become the extra conditions on the pattern of zeros. We have numerically solved all those conditions on the pattern of zeros. In this section, we list some of the numerical solutions.

We like to point out that the patterns of zeros for FQH states described by simple-current vertex algebra do not have the additive property. This is because given two FQH wave functions described by simple-current vertex algebra, their product in general cannot be described any simple-current vertex algebra. The direct product of two simple-current vertex algebra, in general, contains at least one dimension-2 primary field of Virasoro algebra that violates the Abelian fusion algebra. Thus the direct product of two simple-current vertex algebra is not a simple-current vertex algebra in general.

Among many solutions of the consistent conditions are the Z_n parafermion algebras, which are the simplest simple-current vertex algebra. The Z_n parafermion algebras give rise to Z_n parafermion wave functions Φ_{Z_n}. As an example of no additive property, the pattern of zeros for the product wave function $\Phi_{Z_2\otimes Z_3}\equiv\Phi_{Z_2}\Phi_{Z_3}$ does not satisfy the consistent conditions for the simple-current vertex algebra, indicating that the direct product of Z_2 and Z_3 parafermion vertex algebras is not a simple-current vertex algebra. In the following, we only list some solutions that are not Z_n parafermion algebras.

Z_2 simple-current vertex algebra,

$$n=2:\quad c=1\quad (Z_2|Z_2\text{ state}),$$

$$\{m;h_1^{\rm sc}\cdots h_{n-1}^{\rm sc}\}=\{4;1\},$$

$$\{p;M_1\cdots M_{n-1}\}=\{2;0\},$$

$$\{n_0\cdots n_{m-1}\}=\{2\ 0\ 0\ 0\}. \tag{124}$$

$$
\begin{aligned}
& n=2: \quad (Z_2|Z_2|Z_2 \text{ state}), \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \left\{6;\frac{3}{2}\right\}, \\
& \{p;M_1\cdots M_{n-1}\} = \{3;0\}, \\
& \{n_0\cdots n_{m-1}\} = \{2\ 0\ 0\ 0\ 0\ 0\}.
\end{aligned} \tag{125}
$$

Z_3 simple-current vertex algebra,

$$
\begin{aligned}
& n=3: \quad (Z_3|Z_3 \text{ state}), \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \left\{4;\frac{4}{3}\ \frac{4}{3}\right\}, \\
& \{p;M_1\cdots M_{n-1}\} = \{2;0\ 0\}, \\
& \{n_0\cdots n_{m-1}\} = \{3\ 0\ 0\ 0\}.
\end{aligned} \tag{126}
$$

Z_4 simple-current vertex algebra,

$$
\begin{aligned}
& n=4: \quad c=1 \quad (C_4 \text{ state}), \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \{4;1\ 1\ 1\}, \\
& \{p;M_1\cdots M_{n-1}\} = \left\{2;\frac{1}{2}\ 1\ \frac{1}{2}\right\}, \\
& \{n_0\cdots n_{m-1}\} = \{2\ 2\ 0\ 0\}.
\end{aligned} \tag{127}
$$

$$
\begin{aligned}
& n=4: \quad c=1 \quad (Z_4|Z_2 \text{ state}), \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \left\{6;\frac{5}{4}\ 1\ \frac{5}{4}\right\}, \\
& \{p;M_1\cdots M_{n-1}\} = \{3;1\ 2\ 1\}, \\
& \{n_0\cdots n_{m-1}\} = \{2\ 0\ 2\ 0\ 0\ 0\}.
\end{aligned} \tag{128}
$$

$$
\begin{aligned}
& n=4: \text{ (Gaffnian state)}, \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \left\{6;\frac{3}{4}\ 0\ \frac{3}{4}\right\}, \\
& \{p;M_1\cdots M_{n-1}\} = \left\{3;\frac{3}{2}\ 3\ \frac{3}{2}\right\}, \\
& \{n_0\cdots n_{m-1}\} = \{2\ 0\ 0\ 2\ 0\ 0\}.
\end{aligned} \tag{129}
$$

$$
\begin{aligned}
& n=4: \quad (C_4|C_4 \text{ state}), \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \{8;2\ 2\ 2\}, \\
& \{p;M_1\cdots M_{n-1}\} = \{4;1\ 2\ 1\}, \\
& \{n_0\cdots n_{m-1}\} = \{2\ 0\ 2\ 0\ 0\ 0\ 0\ 0\}.
\end{aligned} \tag{130}
$$

Z_6 simple-current vertex algebra,

$$
\begin{aligned}
& n=6: \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \left\{6;\frac{3}{2}\ 2\ \frac{5}{2}\ 2\ \frac{3}{2}\right\}, \\
& \{p;M_1\cdots M_{n-1}\} = \{3;1\ 2\ 2\ 2\ 1\}, \\
& \{n_0\cdots n_{m-1}\} = \{2\ 2\ 2\ 0\ 0\ 0\}.
\end{aligned} \tag{131}
$$

$$
\begin{aligned}
& n=6: \quad c=1, \\
& \{m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\} = \left\{8;\frac{4}{3}\ \frac{4}{3}\ 1\ \frac{4}{3}\ \frac{4}{3}\right\}, \\
& \{p;M_1\cdots M_{n-1}\} = \{3;2\ 4\ 5\ 4\ 2\}, \\
& \{n_0\cdots n_{m-1}\} = \{2\ 1\ 0\ 1\ 2\ 0\ 0\ 0\}.
\end{aligned} \tag{132}
$$

We like to stress that the above pattern of zeros are only checked to satisfy the consistent conditions that do not contain structure constants $C_{a,b}$. It remains to be shown that there exist $C_{a,b}$ for those patterns of zeros that satisfy all the consistent conditions for structure constants (from GJI's). When we check those additional conditions for $C_{a,b}$, we find that the C_4 pattern of zero $\{n;m;h_a^{\text{sc}}\}=\{4;4;1\ 1\ 1\}$ does not correspond to any symmetric polynomial as discussed in Sec. IV E.

We will discuss some other patterns of zeros in detail later. We will show how the central charge c, the structure constants $C_{a,b}$ and the quasiparticle scaling dimension $h_{\gamma+a}^{\text{sc}}$ of the corresponding vertex algebra can be determined from the pattern of zeros $\{n,m;h_a^{\text{sc}}\}$, through the consistent conditions in Secs. V B and VI D and in Appendix F. Those consistent conditions are generated by useful GJI's: (D3) or (D6) with Eq. (D11).

To calculate the central charge and the quasiparticle scaling dimensions $\{c;C_{a,b};h_{\gamma+a}^{\text{sc}}\}$, in the first step we will try to determine them from conditions in Sec. V B, i.e., we do not specify subleading order term (F1) in OPE. If these conditions do not give enough information, then we will resort to more conditions in Appendix F, which is based on the subleading OPE term (F1).

We note that some pattern of zeros can directly fix the central charge, and we list the central charge for those patterns of zeros as in above. The Z_n parafermion patterns of zeros are examples in this class. While for other patterns of zeros, the central charges depend on the structure constants $C_{a,b}$. We will calculate those central charges below. There are even patterns of zeros that do not completely determine the simple-current vertex algebra. We need to include additional information C_{ab} to determine the corresponding simple-current vertex algebra. The $Z_3|Z_3$, $Z_4|Z_2$ states, etc., are examples in this class of pattern of zeros.

We also give names for some patterns of zeros. For example, the $C_n|C_n$ pattern of zeros $\{m;h_1^{sc}\cdots h_{n-1}^{sc}\}=\{2n;2\ 2\cdots 2\}$ is the sum of two C_n pattern of zeros $\{m;h_1^{sc}\cdots h_{n-1}^{sc}\}=\{n;1\ 1\cdots 1\}$. Also, the $Z_4|Z_2$ pattern of zeros is described by $\{m;h_1^{sc}\cdots h_{n-1}^{sc}\}=\{6;\frac{5}{4}\ 1\ \frac{5}{4}\}$ which is a sum of $\{m;h_1^{sc}\cdots h_{2n-1}^{sc}\}=\{4;\frac{1}{2}\ 0\ \frac{1}{2}\}$ for the Z_2 parafermion state and $\{m;h_1^{sc}\cdots h_{n-1}^{sc}\}=\{2;\frac{3}{4}\ 1\ \frac{3}{4}\}$ for the Z_4 parafermion state. (Note that the Z_2 parafermion state is also described by $\{m;h_1^{sc}\cdots h_{n-1}^{sc}\}=\{2;\frac{1}{2}\}$.[11]) However, the wave function of such a $Z_4|Z_2$ state is different from the product of a Z_2 parafermion wave function and a Z_4 parafermion wave function. The product wave function called the $Z_2\otimes Z_4$ state is described by a Z_4 vertex algebra given by the direct product of the Z_2 parafermion algebra and Z_4 parafermion algebra. Such a Z_4 vertex algebra is different from any Z_4 simple-current vertex algebras, featured by an extra dimension-2 primary field. However, both the $Z_4|Z_2$ and $Z_2\otimes Z_4$ states have the same pattern of zeros. This is an example showing that the same pattern of zeros $\{m;h_1^{sc}\cdots h_{n-1}^{sc}\}=\{6;\frac{5}{4}\ 1\ \frac{5}{4}\}$ can correspond to more than one FQH wave functions.

B. Z_n parafermion vertex algebra: Z_n parafermion states with $\{M_k=0;p=1;m=2\}$

In this simplest case we have $p=1$, $M_k=0$. For example, the Z_3 parafermion state is described by the following pattern of zeros:

$$n=3:\quad Z_3\text{ state},$$

$$\{m;h_1^{sc}\cdots h_{n-1}^{sc}\}=\left\{2;\frac{2}{3}\ \frac{2}{3}\right\},$$

$$\{p;M_1\cdots M_{n-1}\}=\{1;0\ 0\},$$

$$\{n_0\cdots n_{m-1}\}=\{3\ 0\}. \tag{133}$$

In general, we have (we do not specify $p=1$ until necessary, trying to obtain some general conclusions on $Z_n|\cdots|Z_n$ series)

$$h_a^{sc}=p\frac{a(n-a)}{n},$$

$$\alpha_{a,b}=\frac{2pab}{n}-2p(a+b-n)\theta(a+b-n). \tag{134}$$

As a result we have

$$\mu_{a,b}=1,\quad C_{a,b}=C_{b,a}. \tag{135}$$

Besides, $d_{a,b}$ defined in Eq. (F4) has a simple form in this case,

$$d_{a,b}\equiv\frac{1}{2}\left(1+\frac{h_a^{sc}-h_b^{sc}}{h_{a+b}^{sc}}\right)=d_{n-a,n-b}=\frac{a}{a+b}\quad\text{if } a+b<n. \tag{136}$$

In Eq. (A18) we have $\Delta M[a,b,c]=0$ and $\Delta_3(a,b,c)$ can only be multiples of $2p$.

At first let's take a look at $\{A,B,C\}=\{\psi_a,\psi_b,\psi_c\}$, $a+b,b+c,a+c\neq 0 \bmod n$. Only when $\Delta_3(a,b,c)=0$ there are extra consistent conditions in Sec. V B 1, i.e., $a+b+c\leqslant n$ or $a+b+c\geqslant 2n$ we have

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}=C_{a,c}C_{b,a+c}. \tag{137}$$

Particularly when $a+b+c=0 \bmod n$ we have

$$C_{a,b}=C_{a,c}=C_{b,c}. \tag{138}$$

The other consistent condition is satisfied by Eq. (136).

For $A=B=C=\psi_{n/2}$, n=even we know that $\Delta_3(n/2,n/2,n/2)=4h_{n/2}^{sc}=pn$. Only when $np/2\leqslant 2$ there are extra consistent conditions in Sec. V B 4, i.e., when $n\leqslant 4$ for $p=1$ and $n\leqslant 2$ for $p=2$.

The above conclusions hold for any $p\in\mathbb{N}$. Now let us enforce $p=1$ for this special series.

For $\{A,B,C\}=\{\psi_a,\psi_b,\psi_{-b}\}$, $a\pm b\neq 0 \bmod n$ we know $\Delta_3(a,b,-b)\geqslant 2p$. Only when $\Delta_3(a,b,-b)=2$ there are extra consistent conditions in Sec. V B 2, i.e., when $a=1<b,n-b<n$, $b=1<a<n-1$ or $1\leqslant n,n-b<a=n-1$.

For $\{A,B,C\}=\{\psi_a,\psi_a,\psi_{-a}\}$, $a\neq n/2 \bmod n$, similarly only when $\Delta_3(a,a,-a)=2,4$ there are consistent conditions in Sec. V B 3, i.e., when $a=1,2$ or $a=n-1,n-2$.

First since $\Delta_3(1,1,-1)=\Delta_3(-1,-1,1)=2$ and $h_1^{sc}=1-1/n$, $\alpha_{1,1}=2/n$ from Sec. V B 3 we have

$$c=\frac{2(n-1)}{n+2},$$

$$C_{1,1}C_{2,n-1}=C_{n-1,n-1}C_{n-2,1}=\frac{2(n-1)}{n}.$$

With central charge c in hand, from $\Delta_3(2,2,-2)=\Delta_3(-2,-2,2)=4$ we have

$$C_{2,2}C_{4,n-2}=C_{n-2,n-2}C_{n-4,2}=\frac{6(n-2)(n-3)}{n(n-1)}. \tag{139}$$

Similarly from $\Delta_3(1,b,-b)=\Delta_3(a,1,-1)=\Delta_3(-1,-b,b)=\Delta_3(-a,-1,1)=2$ we have

$$C_{b,n-1}C_{n-b,b-1}=\frac{b(n+1-b)}{n},$$

$$C_{a,n-1}C_{1,a-1}=\frac{a(n+1-a)}{n}. \tag{140}$$

These are all the extra consistent conditions. Using Eqs. (137) and (138) repeatedly we find out that the independent conditions besides Eqs. (137) and (138) and $C_{a,b}=C_{b,a}$ are just

$$C_{1,a}C_{n-1,n-a}=\frac{(a+1)(n-a)}{n}. \tag{141}$$

Other structure constants can be expressed as

$$C_{a,b}=\frac{C_{a,b-1}C_{1,a+b-1}}{C_{1,b-1}}=\cdots=\frac{\prod_{i=0}^{b-1}C_{1,a+i}}{\prod_{j=1}^{b-1}C_{1,j}}=\frac{\prod_{i=1}^{a+b-1}C_{1,i}}{\prod_{i=1}^{a-1}C_{1,i}\prod_{i=1}^{b-1}C_{1,i}} \tag{142}$$

if $a+b\leqslant n$;

$$C_{a,b}=\frac{C_{a,b+1}C_{n-1,a+b+1}}{C_{n-1,b+1}}=\cdots=\frac{\prod_{i=0}^{n-b-1}C_{n-1,a-i}}{\prod_{j=1}^{n-b-1}C_{n-1,n-j}}=\frac{\prod_{i=1}^{n-a+n-b-1}C_{n-1,n-i}}{\prod_{i=1}^{n-a-1}C_{n-1,n-i}\prod_{i=1}^{n-b-1}C_{n-1,n-i}} \tag{143}$$

if $a+b>n$. Notice that the above two equations are compatible with Eq. (137). Using Eq. (141) we immediately have

$$C_{a,b}C_{n-a,n-b}=\frac{\prod_{i=0}^{b-1}C_{1,a+i}C_{n-1,n-a-i}}{\prod_{j=1}^{b-1}C_{1,j}C_{n-1,n-j}}=\frac{\Gamma(a+b+1)\Gamma(n-a+1)\Gamma(n-b+1)}{\Gamma(n+1)\Gamma(a+1)\Gamma(b+1)\Gamma(n-a-b+1)}, \tag{144}$$

$$\forall\ 1\leqslant a,b<a+b\leqslant n. \tag{145}$$

To summarize, the consistent conditions in Sec. V B determine the central charge and fix the structure constants to the following form:

$$C_{a,b}=\frac{\prod_{i=0}^{b-1}\lambda_{a+i}}{\prod_{j=1}^{b-1}\lambda_j}\sqrt{\frac{\Gamma(a+b+1)\Gamma(n-a+1)\Gamma(n-b+1)}{\Gamma(n+1)\Gamma(a+1)\Gamma(b+1)\Gamma(n-a-b+1)}} \tag{146}$$

if $1\leqslant a,b<a+b\leqslant n$. Free parameters $\{\lambda_a|a=1,\ldots,n-1\}$ are nonzero complex numbers, defined by $C_{1,a}=\lambda_a^2 C_{n-1,n-a}$. Moreover, condition (138) requires that $C_{a,n-a}=1$, so from Eq. (142) we have the following "reflection" condition on $\{\lambda_a\}$:

$$\lambda_{n-1}=\frac{\lambda_{n-2}}{\lambda_1}=\cdots=\frac{\lambda_{n-1-k}}{\lambda_k}=\cdots=1. \tag{147}$$

We point out that the above conclusions are all obtained from conditions in Sec. V B, i.e., we have not introduced the subleading order OPE (F1) and new conditions in Appendix F yet. Now we apply conditions in Appendix F to see whether the normalization constants $\{\lambda_a|a=1,\ldots,[\frac{n-1}{2}]\}$ can be determined or not.

According to Appendix F 2 a, choosing those $\Delta_3(a,b,c)=2p=2$ with $a+b,b+c,a+c\neq 0 \bmod n$, $a+b+c=n+1$ leads to the following new constraints:

$$\lambda_{a-1}\lambda_{n-a}=1, \tag{148}$$

which means that $\lambda_a=\pm 1$. Other useful GJI's like $\Delta_3(a,b,c)=4p=4$ with $a+b,b+c,a+c\neq 0 \bmod n$ does not result in any new constraints. So finally we can conclude that considering the subleading order OPE (F1), the structure of such a Z_n simple-current vertex algebra is determined self-consistently as

$$c=\frac{2(n-1)}{n+2},\quad \lambda_a=\pm 1, \tag{149}$$

$$C_{a,b} = \frac{\prod_{i=0}^{b-1} \lambda_{a+i}}{\prod_{j=1}^{b-1} \lambda_j} \sqrt{\frac{\Gamma(a+b+1)\Gamma(n-a+1)\Gamma(n-b+1)}{\Gamma(n+1)\Gamma(a+1)\Gamma(b+1)\Gamma(n-a-b+1)}}, \tag{150}$$

$$C_{n-a,n-b} = \frac{\prod_{j=1}^{b-1} \lambda_j}{\prod_{i=0}^{b-1} \lambda_{a+i}} \sqrt{\frac{\Gamma(a+b+1)\Gamma(n-a+1)\Gamma(n-b+1)}{\Gamma(n+1)\Gamma(a+1)\Gamma(b+1)\Gamma(n-a-b+1)}}, \quad \forall\ a+b \leqslant n, \tag{151}$$

$$\lambda_{a-1} = \lambda_{n-a}, \quad \lambda_0 = \lambda_{n-1} = 1, \quad \lambda_{a+n} = \lambda_a. \tag{152}$$

It is interesting to see that the Z_n-parafermion pattern of zeros does not completely fix the structure constants $C_{a,b}$. Do those different structure constants $C_{a,b}$ corresponding to different choices of λ_a give rise to different FQH wave functions, even through they all have the same pattern of zeros? In general, the different structure constants (even with the same pattern of zeros) will give rise to different FQH wave functions. But in this particular case, all the above different structure constants for different choices of $\lambda_a = \pm 1$ give rise to the same FQH wave function. So those different structure constants describe the same FQH state.

To see this, let us introduce

$$\tilde{\psi}_a = \chi_a \psi_a, \quad \chi_a = \pm 1, \quad \chi_{a+n} = \chi_a. \tag{153}$$

Those modified simple current operators will generate the same FQH state. But the structure constants of $\tilde{\psi}_a$ is changed,

$$\tilde{C}_{a,b} = C_{a,b} \frac{\chi_a \chi_b}{\chi_{a+b}}. \tag{154}$$

So such kind of change in structure constants,

$$C_{a,b} \to \tilde{C}_{a,b} = C_{a,b} \frac{\chi_a \chi_b}{\chi_{a+b}}, \tag{155}$$

does not generate new FQH wave function. Therefore $C_{a,b}$ and $\tilde{C}_{a,b} = C_{a,b}\frac{\chi_a\chi_b}{\chi_{a+b}}$ describe the same FQH state. We will call transformation (155) an equivalence transformation.

Note that the factors can be rewritten as

$$\frac{\prod_{i=0}^{b-1} \lambda_{a+i}}{\prod_{j=1}^{b-1} \lambda_j} = \frac{\prod_{i=0}^{a+b-1} \lambda_i}{\prod_{i=0}^{a-1} \lambda_i \prod_{i=0}^{b-1} \lambda_i},$$

$$\frac{\prod_{j=1}^{b-1} \lambda_j}{\prod_{i=0}^{b-1} \lambda_{a+i}} = \frac{\prod_{i=0}^{n-b-1} \lambda_{n-a+i}}{\prod_{j=1}^{n-b-1} \lambda_j} = \frac{\prod_{i=0}^{n-a+n-b-1} \lambda_i}{\prod_{i=0}^{n-a-1} \lambda_i \prod_{i=0}^{n-b-1} \lambda_i}, \tag{156}$$

where we have used Eq. (148). So if we choose $\chi_a = \Pi_{i=0}^{a-1} \lambda_i$, the equivalence transformation (155) will remove the λ_a dependent factors in the structure constants. This completes our proof. We see that the Z_n parafermion patterns of zeros, $\{M_k=0; p=1; m=2\}$, completely determine the structure constants C_{ab} and the central charge c.

It is interesting to note that we can use the equivalence transformation (155) to make $C_{a,b}=1$ for all $a+b \leqslant n$, as one can see from Eq. (142). We can also use transformation (155) to make $C_{a,b}=1$ for all $a+b \geqslant n$, as one can see from Eq. (143). But we cannot make all $C_{a,b}=1$.

C. Quasiparticles in the Z_2 parafermion state

In this section, we will study the quasiparticles in the Z_2 parafermion state. For the Z_2 parafermion state, we have a simple current with scaling dimension $h_1^{sc}=1/2$ and central charge $c=1/2$. According to Ref. 13, the patterns of zeros $\{k_{\gamma;a}^{sc}; Q_\gamma\}$ for the quasiparticles are obtained by solving conditions (97) and (98). The result is listed in Table I. There are three types of quasiparticles. In fact, these three quasiparticles belong to two different families. The quasiparticles in the same family can change into each other by combining an Abelian quasiparticle. The two quasiparticles in the first family differ from each other merely by a magnetic translation[12,13] (i.e., by an insertion of an Abelian magnetic flux tube), while the third quasiparticle differs from the first two in their non-Abelian content. For a family of quasiparticles differ by magnetic translations, we only need to study one of them to obtain all the information of simple current part [the difference between different quasiparticles in such a family comes solely from a U(1) factor].

First let us study the third quasiparticle $\{k_{\gamma;1}^{sc}, \ldots, k_{\gamma;n}^{sc}; Q_\gamma\} = \{0, 0; \frac{1}{2}\}$. Using the following relations

TABLE I. The pattern of zeros and the charges Q for the quasiparticles in the Z_2 parafermion state. $n_{\gamma;1}\cdots n_{\gamma,m-1}$ is the occupation sequence characterizing the quasiparticle γ [defined below Eq. (103)]. The quasiparticles are labeled by the index I. The scaling dimensions of the quasiparticle operators are sums of the contributions from the simple-current vertex algebra and the Gaussian model: $h_\gamma = h^{\text{sc}} + h^{\text{ga}}$.

I	I_{na}	$n_{\gamma;0\cdots m-1}$	$nk^{\text{sc}}_{\gamma;1\cdots n}$	Q	$h^{\text{sc}}+h^{\text{ga}}$
0	0_{na}	2 0	1 −1	0	$0+0$
1	0_{na}	0 2	1 −1	1	$0+\frac{1}{2}$
2	1_{na}	1 1	0 0	1/2	$\frac{1}{16}+\frac{1}{8}$

TABLE II. The pattern of zeros and the charges Q for the quasiparticles in the Z_3 parafermion state. The quasiparticles are labeled by the index I. The scaling dimensions of the quasiparticle operators are sums of the contributions from the simple-current vertex algebra and the Gaussian model: $h_\gamma = h^{\text{sc}} + h^{\text{ga}}$.

I	I_{na}	$n_{\gamma;0\cdots m-1}$	$nk^{\text{sc}}_{\gamma;1\cdots n}$	Q	$h^{\text{sc}}+h^{\text{ga}}$
0	0_{na}	3 0	2 0 −2	0	$0+0$
1	0_{na}	0 3	2 0 −2	3/2	$0+\frac{3}{4}$
2	1_{na}	2 1	1 −1 0	1/2	$\frac{1}{15}+\frac{1}{12}$
3	1_{na}	1 2	0 1 −1	1	$\frac{1}{15}+\frac{1}{3}$

derived from Eq. (107) (which will be used frequently in the quasiparticle calculation):

$$\alpha_{a,\gamma+b} = h^{\text{sc}}_a - \sum_{k=1}^{a} k^{\text{sc}}_{\gamma;b+k},$$

$$\Delta_3(a,b,\gamma+c) = \alpha_{a,\gamma+c} + \alpha_{b,\gamma+c} - \alpha_{a+b,\gamma+c}, \qquad (157)$$

we find $\Delta_3(1,1,\gamma)=\Delta_3(1,1,\gamma+1)=1$, $\alpha_{1,\gamma}=\alpha_{1,\gamma+1}=1/2$. So from Sec. VI D 3, we see that

$$h^{\text{sc}}_\gamma = h^{\text{sc}}_{\gamma+1} = 1/16,$$

$$C_{1,\gamma+1} = 1/2. \qquad (158)$$

This is the famous disorder operator of a Z_2 or Ising vertex algebra.

Next we study the first quasiparticle $\{k^{\text{sc}}_{\gamma;1},\ldots,k^{\text{sc}}_{\gamma;n};Q_\gamma\} = \{\frac{1}{2},-\frac{1}{2};0\}$ from its family. With $\Delta_3(1,1,\gamma)=0$ and $\Delta_3(1,1,\gamma+1)=2$ we have from Sec. VI D 3 that

$$h^{\text{sc}}_\gamma = 0, \quad h^{\text{sc}}_{\gamma+1} = 1/2,$$

$$C_{1,\gamma+1} = 1, \quad \partial\sigma_\gamma \equiv 0, \qquad (159)$$

suggesting that σ_γ is proportional to the identity operator. This means that this quasiparticle is simply the trivial vacuum modulo electrons.

We like to stress that for the above two quasiparticles, the structure constants $C_{a,\gamma+b}$ are uniquely determined by the quasiparticles pattern of zeros $\{k^{\text{sc}}_{\gamma;a}\}$. So the quasiparticles in the Z_2 parafermion theory are uniquely described by the quasiparticles pattern of zeros $\{k^{\text{sc}}_{\gamma;a};Q_\gamma\}$. Each index I in Table I label a unique quasiparticle pattern of zeros, and so the index I also label a unique quasiparticle for the Z_2 parafermion state.

We can also obtain the fusion algebra using the method in Ref. 13. We find that

$$0\times 0=0, \quad 0\times 1=1, \quad 0\times 2=2,$$

$$1\times 1=0, \quad 1\times 2=2, \quad 2\times 2=0+1, \qquad (160)$$

where we have used the index I to label different quasiparticles (see Table I). We have regarded two quasiparticles to be equivalent if they differ by some electrons. The index I really label the above equivalent classes of quasiparticles. We may also define a different equivalent class of quasiparticles by regarding two quasiparticles to be equivalent if they differ by some electrons or by some Abelian magnetic flux tubes. Such classes of quasiparticles are characterized by $\{k^{\text{sc}}_{\gamma;1},\ldots,k^{\text{sc}}_{\gamma;n}\}$ up to a cyclic permutation. We introduce an index I_{na} to label those non-Abelian classes of quasiparticles. From the relation between the two sets of indices I and I_{na} as shown in Table I, we can reduce the fusion algebra (160) to a simpler fusion algebra between the non-Abelian classes of quasiparticles,

$$0_{\text{na}}\times 0_{\text{na}}=0_{\text{na}}, \quad 0_{\text{na}}\times 1_{\text{na}}=1_{\text{na}}, \quad 1_{\text{na}}\times 1_{\text{na}}=0_{\text{na}}+0_{\text{na}}. \qquad (161)$$

D. Quasiparticles in the Z_3 parafermion state

The Z_3 simple-current vertex algebra is characterized by

$$h^{\text{sc}}_1 = h^{\text{sc}}_2 = \frac{2}{3}, \quad c = \frac{4}{5},$$

$$C_{1,1} = C_{2,2} = \frac{2}{\sqrt{3}}, \quad C_{1,2} = C_{2,1} = 1, \qquad (162)$$

where we have fixed the normalization factors to be $\lambda_a=1$.

There are two families of quasiparticles obtained from Eqs. (97) and (98) (see Table II). The first family is represented by quasiparticle $\{k^{\text{sc}}_{\gamma;1},\ldots,k^{\text{sc}}_{\gamma;n};Q_\gamma\}=\{\frac{2}{3},0,-\frac{2}{3};0\}$. With $\Delta_3(1,1,\gamma)=\Delta_3(2,2,\gamma)=\Delta_3(1,1,\gamma+1)=\Delta_3(2,2,\gamma+2)=0$, we find that (see Sec. VI D 2 or Appendix F 2 f)

$$C_{1,\gamma+1} = C_{2,\gamma+2} = C_{1,1} = C_{2,2},$$

$$C_{2,\gamma+1} = C_{1,\gamma+2}. \qquad (163)$$

Then with $\Delta_3(1,2,\gamma)=0$ and $\Delta_3(1,2,\gamma+1)=\Delta_3(1,2,\gamma+2)=2$ we have from Sec. VI D 3 that

$$h_\gamma^{sc}=0,\quad h_{\gamma+1}^{sc}=h_{\gamma+2}^{sc}=\frac{2}{3},\quad \partial\sigma_\gamma\equiv 0,$$

$$C_{2,\gamma+1}=C_{1,\gamma+2}=1,\quad C_{1,\gamma+1}C_{2,\gamma+2}=\frac{4}{3}. \tag{164}$$

Therefore this quasiparticle is characterized by

$$h_\gamma^{sc}=0,\quad h_{\gamma+1}^{sc}=h_{\gamma+2}^{sc}=\frac{2}{3},\quad \partial\sigma_\gamma=0,$$

$$C_{2,\gamma+1}=C_{1,\gamma+2}=1,\quad C_{1,\gamma+1}=C_{2,\gamma+2}=\frac{2}{\sqrt{3}}, \tag{165}$$

$\partial\sigma_\gamma=0$ and $h_\gamma^{sc}=0$ imply that the quasiparticle operator σ_γ is a constant operator with scaling dimension 0. Such an operator is the trivial identity operator.

The second family is represented by a quasiparticle with $\{k_{\gamma;1}^{sc},\ldots,k_{\gamma;n}^{sc};Q_\gamma\}=\{\frac{1}{3},-\frac{1}{3},0;\frac{1}{2}\}$. With $\Delta_3(1,1,\gamma)=\Delta_3(2,2,\gamma+2)=0$ and $\Delta_3(1,1,\gamma+1)=\Delta_3(1,1,\gamma+2)=\Delta_3(2,2,\gamma)=\Delta_3(2,2,\gamma+1)=1$, we find that (see Sec. VI D 2 or Appendix F 2 f)

$$C_{1,\gamma+1}=C_{1,1},\quad C_{2,\gamma+2}=C_{2,2}/2,$$

$$C_{1,\gamma+2}=C_{1,1}C_{2,2}/4,\quad C_{2,\gamma+1}=C_{1,1}C_{2,2}/2. \tag{166}$$

Then with $\Delta_3(1,2,\gamma)=\Delta_3(1,2,\gamma+2)=1$ and $\Delta_3(1,2,\gamma+1)=2$, we find that (see Sec. VI D 3)

$$h_\gamma^{sc}=h_{\gamma+2}^{sc}=\frac{1}{15},\quad h_{\gamma+1}^{sc}=\frac{2}{5},$$

$$C_{2,\gamma+1}=C_{1,\gamma+1}C_{2,\gamma+2}=\frac{2}{3},\quad C_{1,\gamma+2}=\frac{1}{3}. \tag{167}$$

This nontrivial quasiparticle is characterized by

$$h_\gamma^{sc}=h_{\gamma+2}^{sc}=\frac{1}{15},\quad h_{\gamma+1}^{sc}=\frac{2}{5},$$

$$C_{2,\gamma+1}=\frac{2}{3},\quad C_{1,\gamma+2}=\frac{1}{3},$$

$$C_{1,\gamma+1}=C_{1,1}=\frac{2}{\sqrt{3}},\quad C_{2,\gamma+2}=C_{2,2}/2=\frac{1}{\sqrt{3}} \tag{168}$$

Again, for the above two quasiparticles, the structure constants $C_{a,\gamma+b}$ are uniquely determined by the quasiparticles pattern of zeros $\{k_{\gamma;a}^{sc}\}$. So the quasiparticles in the Z_3 parafermion theory are uniquely described by the quasiparticles pattern of zeros $\{k_{\gamma;a}^{sc};Q_\gamma\}$. Each index I in Table II label a unique quasiparticle pattern of zeros, and so the index I also label a unique quasiparticle for the Z_3 parafermion state.

The full fusion algebra between the quasiparticles is[13]

$$0\times 0=0,\quad 0\times 1=1,\quad 0\times 2=2,$$

$$0\times 3=3,\quad 1\times 1=0,\quad 1\times 2=3,$$

$$1\times 3=2,\quad 2\times 2=0+3,\quad 2\times 3=1+2,$$

$$3\times 3=0+3. \tag{169}$$

The fusion algebra between the non-Abelian classes of quasiparticles is

$$0_{na}\times 0_{na}=0_{na},\quad 0_{na}\times 1_{na}=1_{na},$$

$$1_{na}\times 1_{na}=0_{na}+1_{na}. \tag{170}$$

E. $Z_n|Z_n$ series: $\{M_k=0;\, p=2\}$

The $Z_n|Z_n$ vertex algebra is called the "second generation" of Z_n parafermion algebra and is studied in Refs. 42–45. In this case we have

$$h_a^{sc}=2\frac{a(n-a)}{n}. \tag{171}$$

As a result we still have Eqs. (135)–(138); therefore Eqs. (142) and (143) still hold.

Apparently in this case the extra conditions in Sec. V B are not enough to determine the full structure of this vertex algebra since now $\Delta_3(a,b,c)$ are multiples of $2p=4$. So we introduce the subleading order OPE (F1) and resort to new conditions in Appendix F.

Since now we have $\Delta_3(1,b,-b)=\Delta_3(-1,b,-b)=\Delta_3(1,a,-1)=2p=4$ from Appendix F 2 c we have

$$C_{1,a}C_{n-1,n-a}=\frac{(a+1)(n-a)}{n^2(n-2)}\times\left[\frac{4a(n-a-1)(n-1)}{c}+(n-2a)(n-2a-2)\right]. \tag{172}$$

Representing the central charge c in terms of a continuous variable g in the following way:

$$c=\frac{4(n-1)g(n+g-1)}{(n+2g-2)(n+2g)} \tag{173}$$

yields an expression of structure constants in terms of g and normalization constants $\{\lambda_a=\lambda_{n-1-a};a=1,\ldots,[\frac{n-1}{2}]\}$, which is totally similar with Z_n parafermion states,

$$C_{1,a}=\lambda_a\sqrt{\frac{(a+1)(n-a)}{n}\frac{(a+g)(n+g-a-1)}{g(n+g-1)}}, \tag{174}$$

$$C_{a,b} = \frac{\prod_{i=1}^{a+b-1} C_{1,i}}{\prod_{j=1}^{b-1} C_{1,j} \prod_{k=1}^{a-1} C_{1,k}} = \frac{\prod_{i=0}^{b-1} \lambda_{a+i}}{\prod_{j=1}^{b-1} \lambda_j} \times \sqrt{\frac{\Gamma(a+b+1)\Gamma(n-a+1)\Gamma(n-b+1)}{\Gamma(n+1)\Gamma(a+1)\Gamma(b+1)\Gamma(n-a-b+1)}} \times \sqrt{\frac{\Gamma(a+b+g)\Gamma(n-a+g)\Gamma(n-b+g)\Gamma(g)}{\Gamma(n+g)\Gamma(a+g)\Gamma(b+g)\Gamma(n-a-b+g)}}, \tag{175}$$

$$C_{n-a,n-b} = \left(\frac{\prod_{j=1}^{b-1} \lambda_j}{\prod_{i=0}^{b-1} \lambda_{a+i}} \right)^2 C_{a,b}, \quad a+b \leqslant n \tag{176}$$

where reflection condition (147) should also be satisfied for the normalization constants λ_a. It is easy to verify that $\Delta_3(a,b,c)=2p=4$ does not result in any new constraints on free parameters $\{g;\lambda_a | a=1,\ldots,[\frac{n-1}{2}]\}$. Therefore the above are all conditions on this $Z_n|Z_n$ series of vertex algebra.

Using the equivalence transformation (155), we can change the normalization constants to $\lambda_a=1$. So only different g in the structure constants give rise to different FQH states. All those different FQH states have that same pattern of zeros, and we need an additional parameter g to completely characterize the FQH state. For the simple ideal Hamiltonian introduced in Refs. 11 and 12, all those different FQH states have a zero energy. In Ref. 20, additional terms are introduced in the Hamiltonian so that only the $Z_3|Z_3$ state with a particular g can be the zero energy states.

F. Quasiparticles in the $Z_2|Z_2$ state

The $Z_2|Z_2$ state is described by the following pattern of zeros:

$$n=2: \quad c=1 \quad (Z_2|Z_2 \text{ state}),$$

$$\{m;h_1^{\mathrm{sc}}\cdots h_{n-1}^{\mathrm{sc}}\}=\{4;1\},$$

$$\{n_0\cdots n_{m-1}\}=\{2\ 0\ 0\ 0\}. \tag{177}$$

Here we have $n=2$, $p=2$, $M_1=0$, and thus $h_1^{\mathrm{sc}}=1$. Since $\Delta_3(1,1,1)=4$ we have according to Sec. V B 4,

$$c=1. \tag{178}$$

There is no free parameter in such a Z_2 simple current vertex algebra.

Now let us turn to the quasiparticles of this state. There are three families of different quasiparticles, and we will discuss them one by one.

The first family has $\{k_{\gamma;1}^{\mathrm{sc}},\ldots,k_{\gamma;n}^{\mathrm{sc}};Q_\gamma\}=\{1,-1;0\}$ as its representative. With $\Delta_3(1,1,\gamma)=0$ and $\Delta_3(1,1,\gamma+1)=4$ we have

TABLE III. The pattern of zeros and the charges Q for the quasiparticles in the $Z_2|Z_2$ parafermion state. The quasiparticles are labeled by the index I. The scaling dimensions of the quasiparticle operators are given by $h_\gamma=h^{\mathrm{sc}}+h^{\mathrm{ga}}$, where $\eta=C_{1,\gamma+1}/2$ is a free real parameter. Note the index $I=8,9$ each actually corresponds to a class of quasiparticles parametrized by a continuous parameter η.

I	I_{na}	$n_{\gamma;0\cdots m-1}$	$nk^{\mathrm{sc}}_{\gamma;1\cdots n}$	Q	$h^{\mathrm{sc}}+h^{\mathrm{ga}}$
0	0_{na}	2 0 0 0	2 −2	0	$0+0$
1	0_{na}	0 2 0 0	2 −2	1/2	$0+\frac{1}{4}$
2	0_{na}	0 0 2 0	2 −2	1	$0+1$
3	0_{na}	0 0 0 2	2 −2	3/2	$0+\frac{9}{4}$
4	1_{na}	1 1 0 0	1 −1	1/4	$\frac{1}{16}+\frac{1}{16}$
5	1_{na}	0 1 1 0	1 −1	3/4	$\frac{1}{16}+\frac{9}{16}$
6	1_{na}	0 0 1 1	1 −1	5/4	$\frac{1}{16}+\frac{25}{16}$
7	1_{na}	1 0 0 1	−1 1	3/4	$\frac{9}{16}+\frac{9}{16}$
8	2_{na}	1 0 1 0	0 0	1/2	$\eta+\frac{1}{4}$
9	2_{na}	0 1 0 1	0 0	1	$\eta+1$

$$h_\gamma^{\mathrm{sc}}=0, \quad \partial\sigma_\gamma \equiv 0, \quad C_{1,\gamma+1,}=1, \tag{179}$$

indicating this quasiparticle is trivial. We also have $h_{\gamma+1}^{\mathrm{sc}} = h_\gamma^{\mathrm{sc}}+k_{\gamma;1}^{\mathrm{sc}}=1$ from Eq. (107).

The second family is represented by $\{k_{\gamma;1}^{\mathrm{sc}},\ldots,k_{\gamma;n}^{\mathrm{sc}};Q_\gamma\} =\{\frac{1}{2},-\frac{1}{2};\frac{1}{4}\}$. With $\Delta_3(1,1,\gamma)=0$ and $\Delta_3(1,1,\gamma+1)=3$, we find that (see Sec. VI D 3)

$$h_\gamma^{\mathrm{sc}}=1/16, \quad h_{\gamma+1}^{\mathrm{sc}}=9/16, \quad C_{1,\gamma+1}=1/2. \tag{180}$$

This is a nontrivial quasiparticle, resembling the one in an Ising vertex algebra, except for the charge being $Q_\gamma=1/4$ rather than $1/2$ in the Ising case.

The third family is represented by $\{k_{\gamma;1}^{\mathrm{sc}},\ldots,k_{\gamma;n}^{\mathrm{sc}};Q_\gamma\} =\{0,0;\frac{1}{2}\}$. With $\Delta_3(1,1,\gamma)=\Delta_3(1,1,\gamma+1)=2$, we find that (see Sec. VI D 3)

$$h_\gamma^{\mathrm{sc}}=h_{\gamma+1}^{\mathrm{sc}}=C_{1,\gamma+1}/2. \tag{181}$$

Remember that the quasiparticles are described by the data $\{k_{\gamma;a}^{\mathrm{sc}};Q_\gamma;C_{a,\gamma+b}\}$. For the first two family of the quasiparticles, the quasiparticle structure constants $C_{a,\gamma+b}$ are uniquely determined by the quasiparticle pattern of zeros $\{k_{\gamma;a}^{\mathrm{sc}};Q_\gamma\}$. In this case, a pattern of zeros correspond to a single type of quasiparticle. For the third family, the pattern of zeros does not fix $C_{a,\gamma+b}$. Therefore the quasiparticles in the third family are labeled by the pattern of zeros $\{k_{\gamma;a}^{\mathrm{sc}};Q_\gamma\}$ and a free parameter $\eta=C_{1,\gamma+1}/2$. So there are infinite types of quasiparticles in the third family. The energy gap for such kind of quasiparticles must vanish at least in the $\eta\to 0$ limit.

We want to mention here that even introducing subleading order OPE terms, $[\psi_a\sigma_{\gamma+b}]_{\alpha_{a,\gamma+b}-1}=C_{a,\gamma+b}d_{a,\gamma+b}\partial\sigma_{\gamma+a+b}$, like we did in Appendix F cannot fix this free parameter here. There are indeed infinite types of quasiparticles in the $Z_2|Z_2$ simple-current FQH state. This suggests that the $Z_2|Z_2$ simple-current FQH state is gapless for the ideal Hamiltonian introduced in Ref. 11.

Using the method in Ref. 13, we obtain the full fusion algebra between the quasiparticles (expressed in terms of the index I in table III),

$$\begin{array}{ccc}
0\times 0=0 & 0\times 1=1 & 0\times 2=2\\
0\times 3=3 & 0\times 4=4 & 0\times 5=5\\
0\times 6=6 & 0\times 7=7 & 0\times 8=8\\
0\times 9=9 & 1\times 1=2 & 1\times 2=3\\
1\times 3=0 & 1\times 4=5 & 1\times 5=6\\
1\times 6=7 & 1\times 7=4 & 1\times 8=9\\
1\times 9=8 & 2\times 2=0 & 2\times 3=1\\
2\times 4=6 & 2\times 5=7 & 2\times 6=4\\
2\times 7=5 & 2\times 8=8 & 2\times 9=9\\
3\times 3=2 & 3\times 4=7 & 3\times 5=4\\
3\times 6=5 & 3\times 7=6 & 3\times 8=9\\
3\times 9=8 & 4\times 4=1+8 & 4\times 5=2+9\\
4\times 6=3+8 & 4\times 7=0+9 & 4\times 8=5+7\\
4\times 9=4+6 & 5\times 5=3+8 & 5\times 6=0+9\\
5\times 7=1+8 & 5\times 8=4+6 & 5\times 9=5+7\\
6\times 6=1+8 & 6\times 7=2+9 & 6\times 8=5+7\\
6\times 9=4+6 & 7\times 7=3+8 & 7\times 8=4+6\\
7\times 9=5+7 & 8\times 8=0+2+9 & 8\times 9=1+3+8\\
 & 9\times 9=0+2+9 &
\end{array}. \tag{182}$$

Here index $I=8$ or $I=9$ does not correspond to a single quasiparticle. They each actually corresponds to a class of quasiparticles parametrized by a continuous parameter η. We can use $(8,\eta)$ and $(9,\eta)$ to uniquely label those quasiparticles. Thus, for example, the fusion rule $8\times 9=1+3+8$ should to interpreted as $(8,\eta)\times(9,\eta')=1+3+(8,\eta'')$ for some η, η', and η''.

The fusion algebra between the non-Abelian classes of quasiparticles is

$$0_{\rm na}\times 0_{\rm na}=0_{\rm na},\quad 0_{\rm na}\times 1_{\rm na}=1_{\rm na},$$

$$0_{\rm na}\times 2_{\rm na}=2_{\rm na},\quad 1_{\rm na}\times 1_{\rm na}=0_{\rm na}+2_{\rm na},$$

$$1_{\rm na}\times 2_{\rm na}=1_{\rm na}+1_{\rm na},\quad 2_{\rm na}\times 2_{\rm na}=0_{\rm na}+0_{\rm na}+2_{\rm na}, \tag{183}$$

where the relation between I and $I_{\rm na}$ is given in Table III.

G. Quasiparticles in the $Z_3|Z_3$ state

The $Z_3|Z_3$ state is described by the following pattern of zeros:

$$n=3:\quad (Z_3|Z_3\text{ state}),$$

$$\{m;h_1^{\rm sc}\cdots h_{n-1}^{\rm sc}\}=\left\{4;\frac{4}{3}\ \frac{4}{3}\right\},$$

$$\{n_0\cdots n_{m-1}\}=\{3\ 0\ 0\ 0\}. \tag{184}$$

Here we have $n=3$, $p=2$, $M_1=M_2=0$, and thus $h_1^{\rm sc}=h_2^{\rm sc}=\frac{4}{3}$. As shown in Sec. VII E we have

$$C_{1,1}C_{2,2}=\frac{4}{9}\left(\frac{8}{c}-1\right) \tag{185}$$

as the only extra consistent condition of this vertex algebra from useful GJI's. We can use two free parameters $\{c,\lambda\}$ to express the structure constants,

$$C_{1,1}=\lambda,\quad C_{2,2}=\frac{4}{9\lambda}\left(\frac{8}{c}-1\right). \tag{186}$$

However, using the equivalence transformation [see Eq. (155)]

$$\psi_1\to\chi\psi_1,\quad \psi_2\to\chi^{-1}\psi_2,\quad \lambda\to\lambda/\chi^3, \tag{187}$$

we can set $\lambda=1$. So the infinite $Z_3|Z_3$ simple-current vertex algebras are parametrized by only a single real number c.

There are five classes of non-Abelian quasiparticles as shown in Table IV. We shall study these five classes one by one in this section. The first class is the trivial one, represented by the data

$$\{k_{\gamma;1}^{\rm sc},\ldots,k_{\gamma;n}^{\rm sc};Q_\gamma\}=\left\{\frac{4}{3},0,-\frac{4}{3};0\right\}. \tag{188}$$

With $\Delta_3(1,1,\gamma)=\Delta_3(1,1,\gamma+1)=\Delta_3(2,2,\gamma)=\Delta_3(2,2,\gamma+2)=0$ we have for the structure constants,

$$C_{1,\gamma+1}=C_{1,1},\quad C_{2,\gamma+2}=C_{2,2},\quad C_{1,\gamma+2}=C_{2,\gamma+1}. \tag{189}$$

Then with $\Delta_3(1,2,\gamma)=0$ we have

$$C_{1,\gamma+2}=C_{2,\gamma+1}=1,\quad h_\gamma^{\rm sc}=0,\quad \partial\sigma_\gamma=0. \tag{190}$$

which dictates that this is a trivial quasiparticle, proportional to the identity operator.

The second class is represented by the data

$$\{k_{\gamma;1}^{\rm sc},\ldots,k_{\gamma;n}^{\rm sc};Q_\gamma\}=\left\{1,-\frac{1}{3},-\frac{2}{3};\frac{1}{4}\right\}. \tag{191}$$

With $\Delta_3(1,1,\gamma)=\Delta_3(2,2,\gamma+2)=0$ and $\Delta_3(1,1,\gamma+1)=\Delta_3(2,2,\gamma)=1$ we have for the structure constants,

$$C_{1,\gamma+1}=C_{1,1},\quad C_{2,\gamma+2}=C_{2,2}/2,\quad C_{2,\gamma+1}=2C_{1,\gamma+2}. \tag{192}$$

Then with $\Delta_3(1,2,\gamma)=1$ and $\Delta_3(1,2,\gamma+2)=3$ we have

$$C_{1,\gamma+2}=\frac{1}{3},\quad C_{2,\gamma+1}=\frac{2}{3},$$

TABLE IV. The pattern of zeros and the charges Q for the quasiparticles in the $Z_3|Z_3$ parafermion state parametrized by $\{c,\lambda\}$. Note that the quasiparticle quantum numbers do not depend on the second parameter λ. The quasiparticles are labeled by the index I. The scaling dimensions of the quasiparticle operators are sums of the contributions from the simple-current vertex algebra and the Gaussian model: $h_\gamma = h^{sc}+h^{ga}$, where $\eta_\pm$ is given by Eq. (200). Note the index $I=16$, 17, 18, and 19 each actually corresponds to a class of quasiparticles parameterized by a continuous parameter η. Similarly the index $I=12$, 13, 14, and 15 each corresponds to two types of quasiparticles parametrized by $\pm$.

I	I_{na}	$n_{\gamma;0\cdots m-1}$	$nk^{sc}_{\gamma;1\cdots n}$	Q	$h^{sc}+h^{ga}$
0	0_{na}	3 0 0 0	4 0 −4	0	$0+0$
1	0_{na}	0 3 0 0	4 0 −4	3/4	$0+\frac{3}{8}$
2	0_{na}	0 0 3 0	4 0 −4	3/2	$0+\frac{3}{2}$
3	0_{na}	0 0 0 3	4 0 −4	9/4	$0+\frac{27}{8}$
4	1_{na}	2 1 0 0	3 −1 −2	1/4	$\frac{c}{24}+\frac{1}{24}$
5	1_{na}	0 2 1 0	3 −1 −2	1	$\frac{c}{24}+\frac{2}{3}$
6	1_{na}	0 0 2 1	3 −1 −2	7/4	$\frac{c}{24}+\frac{49}{24}$
7	1_{na}	1 0 0 2	−2 3 −1	3/2	$(\frac{c}{24}+\frac{2}{3})+\frac{3}{2}$
8	2_{na}	1 2 0 0	2 1 −3	1/2	$\frac{c}{24}+\frac{1}{6}$
9	2_{na}	0 1 2 0	2 1 −3	5/4	$\frac{c}{24}+\frac{25}{24}$
10	2_{na}	0 0 1 2	2 1 −3	2	$\frac{c}{24}+\frac{8}{3}$
11	2_{na}	2 0 0 1	1 −3 2	3/4	$(\frac{c}{24}+\frac{2}{3})+\frac{3}{8}$
12	3_{na}	2 0 1 0	2 −2 0	1/2	$\eta_\pm+\frac{1}{6}$
13	3_{na}	0 2 0 1	2 −2 0	5/4	$\eta_\pm+\frac{25}{24}$
14	3_{na}	1 0 2 0	0 2 −2	1	$\eta_\pm+\frac{2}{3}$
15	3_{na}	0 1 0 2	0 2 −2	7/4	$\eta_\pm+\frac{49}{24}$
16	4_{na}	1 1 1 0	1 0 −1	3/4	$\eta+\frac{3}{8}$
17	4_{na}	0 1 1 1	1 0 −1	3/2	$\eta+\frac{3}{2}$
18	4_{na}	1 0 1 1	−1 1 0	5/4	$(\eta+\frac{1}{3})+\frac{25}{24}$
19	4_{na}	1 1 0 1	0 −1 1	1	$(\eta+\frac{1}{3})+\frac{2}{3}$

$$h^{sc}_\gamma = \frac{c}{24}, \quad h^{sc}_{\gamma+2} = c\frac{1+3C_{1,\gamma+1}C_{2,\gamma+2}}{8} = \frac{c}{24}+\frac{2}{3} \quad (193)$$

after using the structure constants (186). The above results from GJI's are consistent with Eq. (107).

The third class is represented by the data

$$\{k^{sc}_{\gamma,1},\ldots,k^{sc}_{\gamma,n};Q_\gamma\} = \left\{\frac{2}{3},\frac{1}{3},-1;\frac{1}{2}\right\}. \quad (194)$$

With $\Delta_3(1,1,\gamma)=\Delta_3(2,2,\gamma+2)=1$ and $\Delta_3(1,1,\gamma+1)=\Delta_3(2,2,\gamma)=0$ we have for the structure constants,

$$C_{1,\gamma+1} = C_{1,1}/2, \quad C_{2,\gamma+2} = C_{2,2}, \quad C_{1,\gamma+2} = 2C_{2,\gamma+1}. \quad (195)$$

Then with $\Delta_3(1,2,\gamma)=1$ and $\Delta_3(1,2,\gamma+1)=3$ we have

$$C_{1,\gamma+2} = \frac{2}{3}, \quad C_{2,\gamma+1} = \frac{1}{3},$$

$$h^{sc}_\gamma = \frac{c}{24}, \quad h^{sc}_{\gamma+1} = \frac{c}{24}+\frac{2}{3}. \quad (196)$$

where we have used Eq. (186) in calculating $h^{sc}_{\gamma+1}$ as well.

The fourth class is represented by the data

$$\{k^{sc}_{\gamma,1},\ldots,k^{sc}_{\gamma,n};Q_\gamma\} = \left\{\frac{2}{3},-\frac{2}{3},0;\frac{1}{2}\right\}. \quad (197)$$

With $\Delta_3(1,1,\gamma)=\Delta_3(2,2,\gamma+2)=0$ we have for the structure constants,

$$C_{1,\gamma+1} = C_{1,1}, \quad C_{2,\gamma+2}C_{2,\gamma+1} = C_{2,2}C_{1,\gamma+2}. \quad (198)$$

Then with $\Delta_3(1,2,\gamma)=\Delta_3(1,2,\gamma+2)=2$ we have the following consistent conditions:

$$C_{2,\gamma+1} = C_{1,\gamma+2}+\frac{1}{3} = C_{1,\gamma+1}C_{2,\gamma+2},$$

$$h^{sc}_\gamma = h^{sc}_{\gamma+2} = -\frac{c}{12}+\frac{3c}{8}C_{2,\gamma+1}. \quad (199)$$

Solving the above nonlinear equations gives us the structure constants and quasiparticle scaling dimensions,

$$C_{1,\gamma+1} = C_{1,1}, \quad C_{1,\gamma+2} = C_{2,\gamma+1}-\frac{1}{3},$$

$$C_{2,\gamma+1} = -\frac{2}{9}+\frac{16}{9c} \pm \frac{4\sqrt{(c-2)(c-8)}}{9c},$$

$$C_{2,\gamma+2} = C_{2,2}\left(C_{2,\gamma+1}-\frac{1}{3}\right)\Big/ C_{2,\gamma+1},$$

$$h^{sc}_\gamma = h^{sc}_{\gamma+2} = \frac{2}{3}-\frac{c}{6} \pm \frac{\sqrt{(c-2)(c-8)}}{6} \equiv \eta_\pm, \quad (200)$$

where $\pm$ corresponds to two different branches of solutions. Here $c\leqslant 2$ or $c\geqslant 8$ is required to guarantee the scaling dimension h^{sc}_γ to be a real number.

For a $Z_3|Z_3$ simple-current algebra described by a fixed c and a quasiparticle pattern of zeros indexed by $I=12$, 13, 14, or 15, there are two sets of quasiparticle structure constants that satisfy all the consistent conditions for the GJI. This implies that the index $I=12$, 13, 14, and 15 in Table IV each actually corresponds to two types of quasiparticles parametrized by the two sets of structure constants. Those quasiparticles are uniquely labeled by $(I,+)$ and $(I,-)$, $I=12$, 13, 14, and 15. When $c=2$ or $c=8$, then there is only one type of quasiparticle for each $I=12$, 13, 14, and 15.

The fifth class is represented by the data

$$\{k^{\rm sc}_{\gamma;1},\ldots,k^{\rm sc}_{\gamma;n};Q_\gamma\}=\left\{\frac{1}{3},0,-\frac{1}{3};\frac{3}{4}\right\}. \tag{201}$$

With $\Delta_3(1,1,\gamma)=\Delta_3(1,1,\gamma+1)=\Delta_3(2,2,\gamma)=\Delta_3(2,2,\gamma+2)=1$ we have for the structure constants,

$$C_{1,\gamma+1}=C_{1,1}/2,\quad C_{2,\gamma+2}=C_{2,2}/2,\quad C_{1,\gamma+2}=C_{2,\gamma+1}. \tag{202}$$

Then with $\Delta_3(1,2,\gamma)=2$ and $\Delta_3(1,2,\gamma+1)=\Delta_3(1,2,\gamma+2)=3$ we have

$$h^{\rm sc}_\gamma=\frac{3c}{8}C_{1,\gamma+2}\equiv\eta,$$

$$h^{\rm sc}_{\gamma+1}=h^{\rm sc}_{\gamma+2}=\frac{3c}{8}(C_{1,\gamma+2}+C_{1,\gamma+1}C_{2,\gamma+2})+\frac{c}{24}=\frac{3c}{8}C_{1,\gamma+2}+\frac{1}{3}. \tag{203}$$

where we have used Eq. (186). Just like the $Z_2|Z_2$ states, there are infinite sets of quasiparticles structure constants the satisfy the consistent conditions. Those sets of structure constants is parametrized by a single real number $\eta=\frac{3c}{8}C_{1,\gamma+2}=\frac{3c}{8}C_{2,\gamma+1}$. This implies that the index $I=16$, 17, 18, and 19 in Table IV each corresponds to a class of quasiparticles parametrized by a continuous parameter η. Those quasiparticles are uniquely labeled by (I,η), $I=16$, 17, 18, and 19. We see that there are infinite types of quasiparticles in the $Z_3|Z_3$ state, suggesting that the $Z_3|Z_3$ state is gapless for the ideal Hamiltonian introduced in Refs. 11 and 20.

H. $Z_2|Z_2|Z_2$ state

This Z_2 simple-current state is described by the pattern of zeros,

$$n=2:\quad (Z_2|Z_2|Z_2\ \text{state}),$$

$$\{m;h^{\rm sc}_1\cdots h^{\rm sc}_{n-1}\}=\{6;3/2\},$$

$$\{p;M_1\cdots M_{n-1}\}=\{3;0\},$$

$$\{n_0\cdots n_{m-1}\}=\{2\ 0\ 0\ 0\ 0\ 0\}. \tag{204}$$

Since there are no structure constants for a Z_2 vertex algebra after choosing the proper normalization, the only free parameter in this simple-current vertex algebra is the central charge c. However, since $\Delta_3(1,1,1)=6$ in this case, consistent conditions from GJI's cannot fix the central charge according to Sec. V B 4.

Explicit calculations of simple currents correlation functions suggest that the electron wave functions uniquely depends on the central charge c. We like to stress that the $Z_2|Z_2|Z_2$ state provides an interesting example that the vertex algebra is not determined by the structure constants C_{ab} of the leading terms, but by a structure constant of a subleading term.

TABLE V. The pattern of zeros and the charges Q for the quasiparticles in the $Z_2|Z_2|Z_2$ state. The quasiparticles are labeled by the index I.

I	$I_{\rm na}$	$n_{\gamma;0\cdots m-1}$	$nk^{\rm sc}_{\gamma;1\cdots n}$	Q
0	$0_{\rm na}$	2 0 0 0 0 0	3 −3	0
1	$0_{\rm na}$	0 2 0 0 0 0	3 −3	1/3
2	$0_{\rm na}$	0 0 2 0 0 0	3 −3	2/3
3	$0_{\rm na}$	0 0 0 2 0 0	3 −3	1
4	$0_{\rm na}$	0 0 0 0 2 0	3 −3	4/3
5	$0_{\rm na}$	0 0 0 0 0 2	3 −3	5/3
6	$1_{\rm na}$	1 1 0 0 0 0	2 −2	1/6
7	$1_{\rm na}$	0 1 1 0 0 0	2 −2	1/2
8	$1_{\rm na}$	0 0 1 1 0 0	2 −2	5/6
9	$1_{\rm na}$	0 0 0 1 1 0	2 −2	7/6
10	$1_{\rm na}$	0 0 0 0 1 1	2 −2	3/2
11	$1_{\rm na}$	1 0 0 0 0 1	−2 2	5/6
12	$2_{\rm na}$	1 0 1 0 0 0	1 −1	1/3
13	$2_{\rm na}$	0 1 0 1 0 0	1 −1	2/3
14	$2_{\rm na}$	0 0 1 0 1 0	1 −1	1
15	$2_{\rm na}$	0 0 0 1 0 1	1 −1	4/3
16	$2_{\rm na}$	1 0 0 0 1 0	−1 1	2/3
17	$2_{\rm na}$	0 1 0 0 0 1	−1 1	1
18	$3_{\rm na}$	1 0 0 1 0 0	0 0	1/2
19	$3_{\rm na}$	0 1 0 0 1 0	0 0	5/6
20	$3_{\rm na}$	0 0 1 0 0 1	0 0	7/6

In Table V, we list 21 distinct quasiparticle patterns of zeros which give rise to at least 21 different quasiparticles. Those quasiparticles group into four classes of non-Abelian quasiparticles.

I. Gaffnian: A nonunitary Z_4 example

A Z_4 solution $\{m;h^{\rm sc}_1,\ldots,h^{\rm sc}_{n-1}\}=\{6;\frac{3}{4},0,\frac{3}{4}\}$ is called Gaffnian in literature.[34] It has the following commutation factors:

$$\mu_{1,2}=\mu_{1,3}=\mu_{2,3}=-1. \tag{205}$$

Therefore it is a generalized Z_4 simple-current vertex algebra.

With $\Delta_3(2,2,2)=0$ we know from Sec. V B 4 that

$$\partial\psi_2\equiv 0. \tag{206}$$

Since $\Delta_3(1,1,2)=\Delta_3(2,3,3)=0$ we know from Sec. V B 1 that

$$C_{1,1} = C_{1,2} = -C_{2,1},$$

$$C_{3,3} = C_{2,3} = -C_{3,2},$$

$$C_{3,1} = -1, \quad C_{1,3} = 1 \tag{207}$$

With $\Delta_3(1,1,3)=\Delta_3(1,3,3)=3$ we know from Sec. V B 2 that

$$c = \frac{-2(h_1^{sc})^2}{(2h_1^{sc}-3)(h_1^{sc}-2)} = -\frac{3}{5},$$

$$C_{1,1}C_{2,3} = C_{3,3}C_{1,2} = -1. \tag{208}$$

Other useful GJI's with $\Delta_3(1,2,3)=\Delta_3(2,2,3)=\Delta_3(1,2,2)=0$ in Sec. V B 2 and $\Delta_3(1,1,1)=\Delta_3(3,3,3)=3$ in Sec. V B 1 do not produce any new conditions. Further calculations show that even introducing the subleading order OPE (F1) and applying new conditions in Appendix F would not supply any extra conditions. In summary we have

$$c = -\frac{3}{5}, \quad \partial\psi_2 \equiv 0,$$

$$C_{1,1} = C_{1,2} = -C_{2,1} = \lambda \neq 0,$$

$$C_{3,3} = C_{2,3} = -C_{3,2} = -\lambda^{-1},$$

$$C_{1,3} = 1, \quad C_{3,1} = -1 \tag{209}$$

for this Z_4 simple-current vertex algebra, which corresponds to the Gaffnian wave function. Using the equivalence transformation [see Eq. (155)]

$$(\psi_1,\psi_2,\psi_3) \to (\chi\psi_1,\psi_2,\chi^{-1}\psi_3), \quad \lambda \to \lambda\chi^{-2}, \tag{210}$$

we can set $\lambda=1$. So there is only a single Gaffnian wave function.

Gaffnian state $\{m;h_1^{sc},\ldots,h_{n-1}^{sc}\}=\{6;\frac{3}{4},0,\frac{3}{4}\}$ has two families of different quasiparticles according to conditions (97) and (98) (see Table VI). The first family has the following representative: $\{k_{\gamma;1}^{sc},\ldots,k_{\gamma;n}^{sc};Q_\gamma\}=\{\frac{3}{4},-\frac{3}{4},\frac{3}{4},-\frac{3}{4};0\}$. With $\Delta_3(1,1,\gamma)=\Delta_3(1,1,\gamma+2)=\Delta_3(1,2,\gamma+a)=\Delta_3(2,3,\gamma+a)=\Delta_3(3,3,\gamma)=\Delta_3(3,3,\gamma+2)=0$ we obtain all the structure constants from Sec. VI D 2 or Appendix F 2 f,

$$C_{1,\gamma+1} = C_{1,\gamma+2} = -C_{2,\gamma+1} = \lambda,$$

$$C_{1,\gamma+3} = C_{2,\gamma+2} = -C_{3,\gamma+1} = 1,$$

$$C_{2,\gamma+3} = C_{3,\gamma+3} = -C_{3,\gamma+2} = -\lambda^{-1}. \tag{211}$$

With $\Delta_3(1,3,\gamma)=1=\Delta_3(1,3,\gamma+2)$, $\Delta_3(1,3,\gamma+1)=\Delta_3(1,3,\gamma+3)=2$ and $\Delta_3(2,2,\gamma+a)=0$ we have the quasiparticle scaling dimensions from Sec. VI D 3

$$h_\gamma^{sc} = h_{\gamma+2}^{sc} = 0, \quad h_{\gamma+1}^{sc} = h_{\gamma+3}^{sc} = \frac{3}{4},$$

$$\partial\sigma_\gamma = \partial\sigma_{\gamma+2} \equiv 0. \tag{212}$$

Since $C_{a,b}=C_{a,\gamma+b}$ and $\partial\sigma_\gamma=0$ here, we know this quasiparticle σ_γ must be proportional to the identity operator 1.

TABLE VI. The pattern of zeros and the charges Q for the quasiparticles in the Gaffnian state. The quasiparticles are labeled by the index I. The scaling dimensions of the quasiparticle operators are sums of the contributions from the simple-current vertex algebra and the Gaussian model: $h_\gamma=h^{sc}+h^{ga}$.

I	I_{na}	$n_{\gamma;0\cdots m-1}$	$nk_{\gamma;1\cdots n}^{sc}$	Q	$h^{sc}+h^{ga}$
0	0_{na}	2 0 0 2 0 0	3 −3 3 −3	0	$0+0$
1	0_{na}	0 2 0 0 2 0	3 −3 3 −3	2/3	$0+\frac{1}{3}$
2	0_{na}	0 0 2 0 0 2	3 −3 3 −3	4/3	$0+\frac{4}{3}$
3	1_{na}	1 1 0 1 1 0	1 −1 1 −1	1/3	$-\frac{1}{20}+\frac{1}{12}$
4	1_{na}	0 1 1 0 1 1	1 −1 1 −1	1	$-\frac{1}{20}+\frac{3}{4}$
5	1_{na}	1 0 1 1 0 1	−1 1 −1 1	2/3	$\frac{1}{5}+\frac{1}{3}$

The second family has the following representative: $\{k_{\gamma;1}^{sc},\ldots,k_{\gamma;n}^{sc};Q_\gamma\}=\{\frac{1}{4},-\frac{1}{4},\frac{1}{4},-\frac{1}{4};\frac{1}{3}\}$. With $\Delta_3(1,1,\gamma)=\Delta_3(1,1,\gamma+2)=\Delta_3(3,3,\gamma)=\Delta_3(3,3,\gamma+2)=1$ and $\Delta_3(1,2,\gamma+a)=\Delta_3(2,3,\gamma+a)=0$ we obtain all the structure constants from Sec. VI D 2 or Appendix F 2 f,

$$C_{1,\gamma+1} = \lambda/2, \quad C_{3,\gamma+3} = -\lambda^{-1}/2,$$

$$C_{1,\gamma+3} = -C_{3,\gamma+1} = 1/2, \quad C_{2,\gamma+2} = 1,$$

$$C_{1,\gamma+2} = -C_{2,\gamma+1} = \lambda,$$

$$C_{3,\gamma+2} = -C_{2,\gamma+3} = \lambda^{-1}. \tag{213}$$

With $\Delta_3(1,3,\gamma)=\Delta_3(1,3,\gamma+2)=\Delta_3(2,2,\gamma+a)=0$ and $\Delta_3(1,3,\gamma+1)=\Delta_3(1,3,\gamma+3)=3$ we have the quasiparticle scaling dimensions from Sec. VI D 3,

$$h_\gamma^{sc} = h_{\gamma+2}^{sc} = -\frac{1}{20}, \quad h_{\gamma+1}^{sc} = h_{\gamma+3}^{sc} = \frac{1}{5}, \tag{214}$$

and the structure constants are consistent with all the useful GJI's. Apparently this quasiparticle is a nontrivial one.

Using the method in Ref. 13, we obtain the full fusion algebra between the quasiparticles (expressed in terms of the index I in Table VI),

$$\begin{array}{lll} 0\times 0=0 & 0\times 1=1 & 0\times 2=2 \\ 0\times 3=3 & 0\times 4=4 & 0\times 5=5 \\ 1\times 1=2 & 1\times 2=0 & 1\times 3=4 \\ 1\times 4=5 & 1\times 5=3 & 2\times 2=1 \\ 2\times 3=5 & 2\times 4=3 & 2\times 5=4 \\ 3\times 3=1+5 & 3\times 4=2+3 & 3\times 5=0+4 \\ 4\times 4=0+4 & 4\times 5=1+5 & 5\times 5=2+3 \end{array}. \tag{215}$$

The fusion algebra between the non-Abelian classes of quasiparticles is

$$0_{\rm na}\times 0_{\rm na}=0_{\rm na},\quad 0_{\rm na}\times 1_{\rm na}=1_{\rm na},$$

$$1_{\rm na}\times 1_{\rm na}=0_{\rm na}+1_{\rm na}. \tag{216}$$

J. $Z_4|Z_2$ state

This solution $\{m;h_1^{\rm sc},\ldots,h_{n-1}^{\rm sc}\}=\{6;\frac{5}{4},1,\frac{5}{4}\}$ is a direct product of a $n=4$ Pfaffian state $\{m;h_1^{\rm sc},\ldots,h_{n-1}^{\rm sc}\}=\{4;\frac{1}{2},0,\frac{1}{2}\}$ and a Z_4 parafermion state $\{m;h_1^{\rm sc},\ldots,h_{n-1}^{\rm sc}\}=\{2;\frac{3}{2},1,\frac{3}{2}\}$.

In this case we have

$$p=3,\quad M_1=M_3=1,\quad M_2=2. \tag{217}$$

It is easy to verify that $\mu_{i,j}=1$ and thus $C_{i,j}=C_{j,i}$.

From Sec. V B 4 we see that $\Delta_3(2,2,2)=4$ determines the central charge,

$$c=1. \tag{218}$$

Then with $\Delta_3(1,1,2)=\Delta_3(2,3,3)=0$ we know from Sec. V B 1 that

$$C_{1,1}=C_{1,2}=C_{2,1},$$

$$C_{3,3}=C_{2,3}=C_{3,2}, \tag{219}$$

$\Delta_3(1,1,3)=\Delta_3(1,3,3)=4$ in Sec. V B 3 and $\Delta_3(1,2,3)=\Delta_3(1,2,2)=\Delta_3(2,2,3)=2$ in Sec. V B 2 both lead to the following conclusions:

$$C_{1,1}C_{2,3}=C_{1,2}C_{3,3}=\frac{5}{2c}=-\frac{5}{8}+\frac{25}{8c}=\frac{5}{2}. \tag{220}$$

Note that $\Delta_3(1,1,1)=\Delta_3(2,2,2)=2$ does not bring us any new constraints. Further studies after introducing subleading order OPE (F1) show that there are no new constraints on the structure constants, so we conclude that

$$c=1,\quad C_{1,3}=C_{3,1}=1,$$

$$C_{1,1}=C_{1,2}=C_{2,1}=\lambda\neq 0,$$

$$C_{2,3}=C_{3,3}=C_{3,2}=\frac{5}{2\lambda} \tag{221}$$

characterizes this Z_4 simple-current vertex algebra. Using the equivalence transformation [see Eq. (155)]

$$(\psi_1,\psi_2,\psi_3)\to(\chi\psi_1,\psi_2,\chi^{-1}\psi_3),\quad \lambda\to\lambda\chi^{-2}, \tag{222}$$

we can set $\lambda=1$. So there is only a single $Z_4|Z_2$ simple-current vertex algebra which correspond to a single FQH wave function.

In Table VII, we list 42 distinct quasiparticle patterns of zeros which give rise to at least 42 different quasiparticles. Those quasiparticles group into eight classes of non-Abelian quasiparticles.

TABLE VII. The pattern of zeros and the charges Q for the quasiparticles in the $Z_4|Z_2$ state. The quasiparticles are labeled by the index I.

I	$I_{\rm na}$	$n_{\gamma;0\cdots m-1}$	Q	I	$I_{\rm na}$	$n_{\gamma;0\cdots m-1}$	Q
0	$0_{\rm na}$	2 0 2 0 0 0	0	1	$0_{\rm na}$	0 2 0 2 0 0	$\frac{2}{3}$
2	$0_{\rm na}$	0 0 2 0 2 0	$\frac{4}{3}$	3	$0_{\rm na}$	0 0 0 2 0 2	2
4	$0_{\rm na}$	2 0 0 0 2 0	$\frac{2}{3}$	5	$0_{\rm na}$	0 2 0 0 0 2	$\frac{4}{3}$
6	$1_{\rm na}$	2 0 1 1 0 0	$\frac{1}{6}$	7	$1_{\rm na}$	0 2 0 1 1 0	$\frac{5}{6}$
8	$1_{\rm na}$	0 0 2 0 1 1	$\frac{3}{2}$	9	$1_{\rm na}$	1 0 0 2 0 1	$\frac{7}{6}$
10	$1_{\rm na}$	1 1 0 0 2 0	$\frac{5}{6}$	11	$1_{\rm na}$	0 1 1 0 0 2	$\frac{3}{2}$
12	$2_{\rm na}$	1 1 1 1 0 0	$\frac{1}{3}$	13	$2_{\rm na}$	0 1 1 1 1 0	1
14	$2_{\rm na}$	0 0 1 1 1 1	$\frac{5}{3}$	15	$2_{\rm na}$	1 0 0 1 1 1	$\frac{4}{3}$
16	$2_{\rm na}$	1 1 0 0 1 1	1	17	$2_{\rm na}$	1 1 1 0 0 1	$\frac{2}{3}$
18	$3_{\rm na}$	2 0 0 2 0 0	$\frac{1}{3}$	19	$3_{\rm na}$	0 2 0 0 2 0	1
20	$3_{\rm na}$	0 0 2 0 0 2	$\frac{5}{3}$				
21	$4_{\rm na}$	1 1 0 2 0 0	$\frac{1}{2}$	22	$4_{\rm na}$	0 1 1 0 2 0	$\frac{7}{6}$
23	$4_{\rm na}$	0 0 1 1 0 2	$\frac{11}{6}$	24	$4_{\rm na}$	2 0 0 1 1 0	$\frac{1}{2}$
25	$4_{\rm na}$	0 2 0 0 1 1	$\frac{7}{6}$	26	$4_{\rm na}$	1 0 2 0 0 1	$\frac{5}{6}$
27	$5_{\rm na}$	2 0 1 0 1 0	$\frac{1}{3}$	28	$5_{\rm na}$	0 2 0 1 0 1	1
29	$5_{\rm na}$	1 0 2 0 1 0	$\frac{2}{3}$	30	$5_{\rm na}$	0 1 0 2 0 1	$\frac{4}{3}$
31	$5_{\rm na}$	1 0 1 0 2 0	1	32	$5_{\rm na}$	0 1 0 1 0 2	$\frac{5}{3}$
33	$6_{\rm na}$	1 1 1 0 1 0	$\frac{1}{2}$	34	$6_{\rm na}$	0 1 1 1 0 1	$\frac{7}{6}$
35	$6_{\rm na}$	1 0 1 1 1 0	$\frac{5}{6}$	36	$6_{\rm na}$	0 1 0 1 1 1	$\frac{3}{2}$
37	$6_{\rm na}$	1 0 1 0 1 1	$\frac{7}{6}$	38	$6_{\rm na}$	1 1 0 1 0 1	$\frac{5}{6}$
39	$7_{\rm na}$	1 1 0 1 1 0	$\frac{2}{3}$	40	$7_{\rm na}$	0 1 1 0 1 1	$\frac{4}{3}$
41	$7_{\rm na}$	1 0 1 1 0 1	1				

K. $C_n|C_n$ series with $\{m;h_1^{\rm sc},\ldots,h_{n-1}^{\rm sc}\}=\{2n;2,\ldots,2\}$

This corresponds to a series of FQH states with filling fraction $\nu=1/2$ for bosonic electrons (and $\nu=1/3$ for fermionic electrons). A $C_4|C_4$ example is given in Eq. (130).

First, from Eq. (E8) we know that $\mu_{a,b}=1$ for such a $C_n|C_n$ simple-current vertex algebra, since all the simple current scaling dimensions are even integers and so are all $\alpha_{a,b}$, $\forall a,b$. As a result we have

$$C_{a,b}=C_{b,a},\quad \forall\, a,b\in\mathbb{Z}. \tag{223}$$

It is straightforward to check that if we do not have the subleading term (F1) in OPE, this solution only has the following extra consistent conditions shown in Section V B 1 with $\Delta_3(a,b,-a-b)=0$, $a,b,a+b\neq 0 \bmod n$,

$$C_{a,b}=C_{a,-a-b}=C_{b,-a-b}, \tag{224}$$

which for sure can be satisfied for all $a,b\in\mathbb{Z}$.

Now we introduce the subleading OPE term (F1) and the new consistent conditions in Appendix F to see whether they are satisfied for this vertex algebra. Note that here we have

$$\alpha_{a,b}=\begin{cases}2, & a+b\neq 0 \bmod n\\ 4, & a+b=0 \bmod n\end{cases} \tag{225}$$

for any $a,b\neq 0 \bmod n$, and also $d_{a,b}=1/2$, $a+b\neq 0 \bmod n$ from Eq. (F4).

Taking any integers $a,b,c\neq 0 \bmod n$, for this such a Z_n simple current vertex algebra we have $\Delta_3(a,b,c)=2$ for $a+b,b+c,a+c,a+b+c\neq 0 \bmod n$, so according to Appendix F 2 a we have

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}=C_{a,c}C_{b,a+c}. \tag{226}$$

Then all consistent conditions are satisfied without requiring that $\partial\psi_a=0$.

$\Delta_3(a,b,c=-a-b)=0$ for $a+b\neq 0 \bmod n$, so according to Appendix F 2 b we have

$$C_{a,b}=C_{a,-a-b}=C_{b,-a-b}. \tag{227}$$

$\Delta_3(a,b,c=-b)=4$ for $a\pm b\neq 0 \bmod n$, so according to Appendix F 2 c we have

$$C_{a,b}C_{a+b,-b}=C_{a,-b}C_{b,a-b}=\frac{8}{c}. \tag{228}$$

$\Delta_3(a,b=a,c=-a)=6$ for $2a\neq 0 \bmod n$, so according to Appendix F 2 d we have

$$d_{a,a}=1/2, \tag{229}$$

which is consistent with Eq. (F4).

$\Delta_3(a=n/2,b=n/2,c=n/2)=8$ for n=even, so according to Sec. V B 4 there are no extra consistent conditions.

In summary, this series of solutions $\{m;h_1^{\text{sc}},\ldots,h_{n-1}^{\text{sc}}\}$ $=\{2n;2,\ldots,2\}$ corresponds to a Z_n simple-current vertex algebra satisfying the following consistent conditions:

$$C_{a,b}=C_{b,a},\quad \forall\, a,b\in\mathbb{Z}, \tag{230}$$

$$C_{a,b}C_{a+b,c}=C_{a,c}C_{b,a+c}=C_{b,c}C_{a,b+c},$$

$$\text{if } a+b,\quad b+c,\quad a+c\neq 0 \bmod n, \tag{231}$$

$$C_{a,b}C_{a+b,\ -b}=C_{a,-b}C_{b,a-b}=\frac{8}{c}\quad \text{if } a\pm b\neq 0 \bmod n. \tag{232}$$

By solving the above conditions in the similarly way as with the Z_n and the $Z_n|Z_n$ series, we find that

$$C_{a,b}=\frac{\prod_{i=1}^{a+b-1}\lambda_i}{\prod_{i=1}^{a-1}\lambda_i\prod_{j=1}^{b-1}\lambda_j}\sqrt{\frac{8}{c}},\quad a+b<n, \tag{233}$$

$$C_{n-a,n-b}=\frac{\prod_{i=1}^{a-1}\lambda_i\prod_{j=1}^{b-1}\lambda_j}{\prod_{i=1}^{a+b-1}\lambda_i}\sqrt{\frac{8}{c}},\quad a+b<n, \tag{234}$$

$$C_{a,n-a}=1, \tag{235}$$

where nonzero complex parameters $\{\lambda_i|i=1,2,\ldots,n-2\}$ satisfy the following constraint:

$$\lambda_{a-1}=\lambda_{n-a},\quad 1\leqslant a\leqslant n-1,\quad \lambda_0=1. \tag{236}$$

If we choose $\chi_a=\prod_{i=0}^{a-1}\lambda_i$, the equivalence transformation (155) will remove the λ_a dependent factors in the structure constants. We find that the $C_n|C_n$ series is characterized by the following data:

$$C_{a,b}=\sqrt{\frac{8}{c}},\quad a+b<n \text{ or } a+b>n, \tag{237}$$

$$C_{a,n-a}=1. \tag{238}$$

Therefore this theory has one free parameters c if $n>2$.

In Table VIII, we list 56 distinct quasiparticle patterns of zeros which give rise to at least 56 different quasiparticles for the $C_3|C_3$ state (or $Z_3|Z_3|Z_3$ state). Those quasiparticles group into ten classes of non-Abelian quasiparticles.

VIII. SUMMARY

The pattern-of-zeros is a powerful way to characterize FQH states.[11–13] However, the pattern-of-zeros approach is not quite complete. It is known that some patterns of zeros do not uniquely describe the FQH states. As a result, we cannot obtain all the topological properties of FQH states from the data of pattern of zeros $\{n;m;S_a\}$.

In this paper, we combine the pattern-of-zero approach with the vertex algebra approach. We find that we can generalize the data of pattern of zeros $\{n;m;S_a\}$ to $\{n;m;S_a;c\}$ to completely describe a FQH state, at least for the many examples discussed in this paper. Many consistent conditions on the new set of data $\{n;m;S_a;c\}$ are obtained from the GJI of the simple-current vertex algebra. Those consistent conditions are sufficient: i.e., if the data $\{n;m;S_a;c\}$ satisfy those conditions, then the data will define a Z_n simple-current vertex algebra and a FQH wave function. Using the new characterization scheme and the Z_n simple-current vertex algebra, we can calculate quasiparticle scaling dimensions, fractional statistics, the central charge of the edge states, as well as many other properties, from the data $\{n;m;S_a;c\}$.

For example, for the Z_2 parafermion state characterized by pattern of zeros $\{n;m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\{2;2;\frac{1}{2}\}$, we find the

TABLE VIII. The pattern of zeros and the charges Q for the quasiparticles in the $C_3|C_3$ state (which also the $Z_3|Z_3|Z_3$ state). The quasiparticles are labeled by the index I.

I	I_{na}	$n_{\gamma;0\cdots m-1}$	Q	I	I_{na}	$n_{\gamma;0\cdots m-1}$	Q
0	0_{na}	3 0 0 0 0 0	0	1	0_{na}	0 3 0 0 0 0	1/2
2	0_{na}	0 0 3 0 0 0	1	3	0_{na}	0 0 0 3 0 0	3/2
4	0_{na}	0 0 0 0 3 0	2	5	0_{na}	0 0 0 0 0 3	5/2
6	1_{na}	2 1 0 0 0 0	1/6	7	1_{na}	0 2 1 0 0 0	2/3
8	1_{na}	0 0 2 1 0 0	7/6	9	1_{na}	0 0 0 2 1 0	5/3
10	1_{na}	0 0 0 0 2 1	13/6	11	1_{na}	1 0 0 0 0 2	5/3
12	2_{na}	1 2 0 0 0 0	1/3	13	2_{na}	0 1 2 0 0 0	5/6
14	2_{na}	0 0 1 2 0 0	4/3	15	2_{na}	0 0 0 1 2 0	11/6
16	2_{na}	0 0 0 0 1 2	7/3	17	2_{na}	2 0 0 0 0 1	5/6
18	3_{na}	2 0 1 0 0 0	1/3	19	3_{na}	0 2 0 1 0 0	5/6
20	3_{na}	0 0 2 0 1 0	4/3	21	3_{na}	0 0 0 2 0 1	11/6
22	3_{na}	1 0 0 0 2 0	4/3	23	3_{na}	0 1 0 0 0 2	11/6
24	4_{na}	1 1 1 0 0 0	1/2	25	4_{na}	0 1 1 1 0 0	1
26	4_{na}	0 0 1 1 1 0	3/2	27	4_{na}	0 0 0 1 1 1	2
28	4_{na}	1 0 0 0 1 1	3/2	29	4_{na}	1 1 0 0 0 1	1
30	5_{na}	1 0 2 0 0 0	2/3	31	5_{na}	0 1 0 2 0 0	7/6
32	5_{na}	0 0 1 0 2 0	5/3	33	5_{na}	0 0 0 1 0 2	13/6
34	5_{na}	2 0 0 0 1 0	2/3	35	5_{na}	0 2 0 0 0 1	7/6
36	6_{na}	2 0 0 1 0 0	1/2	37	6_{na}	0 2 0 0 1 0	1
38	6_{na}	0 0 2 0 0 1	3/2	39	6_{na}	1 0 0 2 0 0	1
40	6_{na}	0 1 0 0 2 0	3/2	41	6_{na}	0 0 1 0 0 2	2
42	7_{na}	1 1 0 1 0 0	2/3	43	7_{na}	0 1 1 0 1 0	7/6
44	7_{na}	0 0 1 1 0 1	5/3	45	7_{na}	1 0 0 1 1 0	7/6
46	7_{na}	0 1 0 0 1 1	5/3	47	7_{na}	1 0 1 0 0 1	7/6
48	8_{na}	1 0 1 1 0 0	5/6	49	8_{na}	0 1 0 1 1 0	4/3
50	8_{na}	0 0 1 0 1 1	11/6	51	8_{na}	1 0 0 1 0 1	4/3
52	8_{na}	1 1 0 0 1 0	5/6	53	8_{na}	0 1 1 0 0 1	4/3
54	9_{na}	1 0 1 0 1 0	1	55	9_{na}	0 1 0 1 0 1	3/2

well-known scaling dimensions (the non-Abelian part) 0, $\frac{1}{2}$, and $\frac{1}{16}$ for the three kind of quasiparticles. For the $Z_2|Z_2$ state characterized by pattern of zeros $\{n;m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\{2;4;1\}$, we find the scaling dimensions and the charges for all its quasiparticles (see Table III). We find that the FQH state described by the $Z_2|Z_2$ simple-current vertex algebra contains infinite types of quasiparticles and two classes of them are parametrized by a real parameter. This indicates that the $Z_2|Z_2$ state is gapless for the ideal Hamiltonian introduced in Ref. 11.

We also studied the $Z_3|Z_3$ state described by the $Z_3|Z_3$ simple-current vertex algebra, with the pattern of zeros $\{n;m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\{3;4;\frac{4}{3}\ \frac{4}{3}\}$. Such a state is also studied in Ref. 20. We show that the $Z_3|Z_3$ state cannot be completely characterized by the pattern-of-zeros data $\{n;m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}}\}=\{3;4;\frac{4}{3}\ \frac{4}{3}\}$. We need to add one more parameter c and use the expanded data $\{n;m;h_1^{\text{sc}}\cdots h_{n-1}^{\text{sc}};c\}=\{3;4;\frac{4}{3}\ \frac{4}{3};c\}$ to completely characterize the $Z_3|Z_3$ state. We find the scaling dimensions and the charges for all its quasiparticles (see Table IV). Again there are infinite types of quasiparticles and four classes of them are parametrized by a real parameter. This again suggests that the $Z_3|Z_3$ state is gapless for the ideal Hamiltonian introduced in Refs. 11 and 20.

The study in this paper is based on the Z_n simple-current vertex algebra. But the Z_n simple-current vertex algebra makes some unnecessary assumptions. It is much more natural to study FQH state based on the more general Z_n vertex algebra. This will be a direction of future exploration.

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APPENDIX A: OTHER WAYS TO LABEL THE PATTERN OF ZEROS

In Sec. II, we have discussed two ways to label the pattern of zeros, one in terms of $\{n;m;S_a\}$ and the other in terms of $\{n;m;h_a^{\text{sc}}\}$. In this section, we will introduce two other more efficient ways to label pattern of zeros. The new ways of labeling automatically satisfy more self consistent conditions.

1. Label the pattern of zeros by a set of non-negative integers $\{a_j\}$

Since $\Delta_3(a,b,c)$ in Eq. (29) is just a linear combination of the h_a^{sc}, there are only $n-1$ independent equations among all the possible n^3 choices of (i,j,k) in Eq. (22). A convenient choice would be $(i,j,k)=(1,1,a)$, $a=1,2,\cdots,n-1$. These equations are

$$\Delta_3(1,1,1) = (2h_1^{\text{sc}} - h_2^{\text{sc}}) + h_1^{\text{sc}} - 2h_2^{\text{sc}} + h_3^{\text{sc}} = a_1 \in \mathbb{N},$$

$$\cdots,$$

$$\Delta_3(1,1,j) = (2h_1^{\text{sc}} - h_2^{\text{sc}}) + h_j^{\text{sc}} - 2h_{j+1}^{\text{sc}} + h_{j+2}^{\text{sc}} = a_j \in \mathbb{N},$$

$$\cdots,$$

$$\Delta_3(1,1,n-3) = (2h_1^{\text{sc}} - h_2^{\text{sc}}) + h_{n-3}^{\text{sc}} - 2h_{n-2}^{\text{sc}} + h_{n-1}^{\text{sc}} = a_{n-3} \in \mathbb{N},$$

$$\Delta_3(1,1,n-2) = (2h_1^{\text{sc}} - h_2^{\text{sc}}) + h_{n-2}^{\text{sc}} - 2h_{n-1}^{\text{sc}} = a_{n-2} \in \mathbb{N},$$

$$\Delta_3(1,1,n-1) = (2h_1^{\text{sc}} - h_2^{\text{sc}}) + h_{n-1}^{\text{sc}} + h_1^{\text{sc}} = a_{n-1} \in \mathbb{N},$$

Here we only used the $h_n^{\text{sc}}=h_0^{\text{sc}}=0$ condition. Adding up these equations together we immediately obtain the following equation:

$$n(2h_1^{\mathrm{sc}} - h_2^{\mathrm{sc}}) = \sum_{i=1}^{n-1} a_i. \tag{A1}$$

By defining another vector $\{A_j\}$,

$$A_j = a_j - (2h_1^{\mathrm{sc}} - h_2^{\mathrm{sc}}) = a_j - \frac{1}{n}\sum_{i=1}^{n-1} a_i, \tag{A2}$$

we have a simple relation

$$\mathbf{X} \cdot \mathbf{h}^{\mathrm{sc}} = \mathbf{A}, \tag{A3}$$

where $\mathbf{h}^{\mathrm{sc}}=(h_1^{\mathrm{sc}},\cdots,h_{n-1}^{\mathrm{sc}})^T$ and $\mathbf{A}=(A_1,\cdots,A_{n-1})^T$ are column vectors and the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \cdots & & & & & \\ \cdots & 0 & 0 & 1 & -2 & 1 \\ \cdots & 0 & 0 & 0 & 1 & -2 \\ 1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix}_{(n-1)\times(n-1)}. \tag{A4}$$

It would be straightforward to check that the above matrix is not singular and its inverse equals

$$n \cdot \mathbf{X}_{i,j}^{-1} = \begin{cases} ij - jn, & j < i \\ ij - (i-1)n, & j \geqslant i. \end{cases} \tag{A5}$$

So we can express $(h_1^{\mathrm{sc}},\ldots,h_{n-1}^{\mathrm{sc}})$ in terms of the $(a_1,\ldots,a_{n-1})$,

$$\begin{aligned} h_a^{\mathrm{sc}} &= \sum_{b=1}^{n-1} \mathbf{X}_{a,b}^{-1} A_b \\ &= \frac{a(n-a)}{2n}\sum_{j=1}^{n-1} a_j + \frac{a-n}{n}\sum_{j=1}^{n-1}(j+1)a_j + \sum_{j=a}^{n-1}(j+1-a)a_j. \end{aligned} \tag{A6}$$

Since the date $\{n;m;h_1^{\mathrm{sc}},\ldots,h_{n-1}^{\mathrm{sc}}\}$ and $\{n;m;a_1,\ldots,a_{n-1}\}$ have an one-to-one correspondence, we can also use $\{n;m;a_1,\ldots,a_{n-1}\}$ to label the pattern of zeros. The $\{n;m;a_1,\ldots,a_{n-1}\}$ labeling scheme is more efficient: once we choose non-negative integers $\{a_1,\ldots,a_{n-1}\}$, we generate h_a^{sc} that already satisfy a part of Eq. (29).

From the reflection conditions (30) of $\{h_a^{\mathrm{sc}}\}$, we can obtain similar reflection conditions for a_j,

$$a_j = a_{n-2-j} \quad (1 \leqslant j \leqslant n-3), \quad a_{n-2} = 0, \tag{A7}$$

then the independent sequence of non-negative integers is actually $\{a_j.1 \leqslant j \leqslant [\frac{n}{2}]-1;a_{n-1}\}$, which contain $[\frac{n}{2}]$ integers. (We use $[x]$ to denote the biggest integer no larger than x.) The $\{a_j\}$ label of the patterns of zeros provides us an efficient way to numerically find the solutions of Eqs. (25) and (27)–(29).

2. Label patterns of zeros by $\{M_k;p;m\}$

a. $\{M_k;p;m\}$ labeling scheme

Using reflection conditions (A7) we can define

$$p \equiv \frac{1}{2}\sum_{j=1}^{n-1} a_j = \frac{a_{n-1}}{2} + \sum_{j=1}^{[n/2]-2} a_j + \begin{cases} a_{[n/2]-1}, & n = \text{odd} \\ a_{[n/2]-1}/2, & n = \text{even}, \end{cases} \tag{A8}$$

and

$$M_k \equiv \frac{n-k}{n}\sum_{j=1}^{n-1}(j+1)a_j - \sum_{j=k}^{n-1}(j+1-k)a_j. \tag{A9}$$

It is easy to verify the following reflection condition for $\{M_k\}$:

$$\text{for } k = 1,2,\ldots,\left[\frac{n}{2}\right]:$$

$$\begin{aligned} M_k &= M_{n-k} \\ &= \sum_{j=1}^{k-2}(j+1)a_j + k\sum_{j=k-1}^{[n/2]-2} a_j + k\cdot\begin{cases} a_{[n/2]-1}, & n = \text{odd} \\ a_{[n/2]-1}/2, & n = \text{even}, \end{cases} \end{aligned} \tag{A10}$$

and another important relation

$$M_2 = 2M_1. \tag{A11}$$

Then we can express $\{h_a^{\mathrm{sc}}\}$ in terms of this new set of independent variables $\{M_k,k=1,3,4,\cdots,[\frac{n}{2}];p\}$ (also $[n/2]$ dimensional),

$$h_a^{\mathrm{sc}} = p\frac{a(n-a)}{n} - M_a. \tag{A12}$$

From definitions we see that both p and $\{M_k,\ k=\text{odd}\}$ can be half-integers, while $\{M_k,\ k=\text{even}\}$ must be integers. When $n=\text{odd}$, $\{M_k,\ k=\text{odd}\}$ must be integers too. In fact, the simplest parafermion vertex algebra[35] (which describes the Z_n parafermion states[17] in a FQH context) corresponds to the case in which $p=1$, $M_k=0$.

Certainly, not all possible choices of $\{M_k;p;m\}$ correspond to valid patterns of zeros. Only those that satisfy conditions (27)–(29), are valid. But $\{M_k;p;m\}$ labeling scheme is an efficient way to generate the valid patterns of zeros.

Now we have two $[\frac{n}{2}]$-dimensional vectors describing the $\{h_a^{\mathrm{sc}}\}$: $\{a_j,j=1,2,\cdots,[\frac{n}{2}]-1;a_{n-1}\}$ and $\{M_k,k=1,3,4,\cdots,[\frac{n}{2}];p\}$. The latter is expressed in terms of the former in Eqs. (A8) and (A10). Conversely, we can express the former in terms of the latter in the following way:

$$a_j = -M_j + 2M_{j+1} - M_{j+2}, \quad j = 1,2,\cdots,\left[\frac{n}{2}\right]-2,$$

$$a_{[n/2]-1} = (M_{[n/2]} - M_{[n/2]-1})\cdot\begin{cases} 1, & n = \text{odd} \\ 2, & n = \text{even}, \end{cases}$$

$$a_{n-1} = 2p - 2M_1. \tag{A13}$$

The $\{M_k;p;m\}$ label of the pattern of zeros has a close tie to simple parafermion CFT.

b. Consistent conditions on $\{M_k;p;m\}$

Now we use this new labeling scheme in Eq. (A12) to see what are the constraints on $\{M_k;p;m\}$ from all the consistent conditions (27)–(29) on $\{h_a^{\text{sc}};m\}$.

At first, with $M_2=2M_1$ (27) leads to

$$S_2 = D_{1,1} = \frac{m-2p}{n} = \text{even}. \tag{A14}$$

Therefore we have

$$m = 2p + 2nk_m, \quad k_m \in \mathbb{N}. \tag{A15}$$

This determines the electron filling fraction $\nu_e=(1+m/n)^{-1}=n/2p+(2k_m+1)n$.

To guarantee mn=even with Eq. (A15) we have another condition on p,

$$p \in \mathbb{N} \quad \text{if } n = \text{odd}. \tag{A16}$$

Since $S_a=\Sigma_{i=2}^{a-1}D_{i,1}$, Eq. (28) naturally guarantees $S_a \in \mathbb{N}$. Moreover we have $D_{a+n,b}=D_{a,b}+mb$, thus we only need to satisfy the following conditions for Eq. (28):

$$D_{a,b} = \begin{cases} M_a + M_b - M_{a+b} + 2k_m ab, & 0 \leqslant a,b \leqslant a+b \leqslant n \\ M_a + M_b - M_{a+b} + 2k_m ab + 2p(a+b-n), & 0 < a,b \leqslant n < a+b \end{cases} \in \mathbb{N}. \tag{A17}$$

Since $\Delta_3(i,j,k)$, $0 \leqslant i \leqslant j \leqslant k < n$ should be non-negative integers, M_k and p satisfy some additional conditions as shown in Eq. (29),

$$\begin{aligned} \Delta_3(i,j,k) &= h_i^{\text{sc}} + h_j^{\text{sc}} + h_k^{\text{sc}} + h_{i+j+k}^{\text{sc}} - h_{i+j}^{\text{sc}} - h_{i+k}^{\text{sc}} - h_{j+k}^{\text{sc}} \\ &= \begin{cases} -\Delta M[i,j,k], & i+j+k \leqslant n \\ -\Delta M[i,j,k] + 2p(i+j+k-n), & j+k \leqslant n < i+j+k \leqslant 2n \\ -\Delta M[i,j,k] + 2pi, & i+k \leqslant n < j+k \leqslant i+j+k \leqslant 2n \\ -\Delta M[i,j,k] + 2p(n-k), & i+j \leqslant n < i+k \leqslant j+k \leqslant i+j+k \leqslant 2n \\ -\Delta M[i,j,k] + 2p(2n-i-j-k), & n < i+j \leqslant i+k \leqslant j+k \leqslant i+j+k \leqslant 2n \\ -\Delta M[i,j,k], & n < i+j \leqslant i+k \leqslant j+k \leqslant 2n < i+j+k \end{cases} \in \mathbb{N}, \end{aligned} \tag{A18}$$

where we defined

$$\begin{aligned} \Delta M[i,j,k] = M_i + M_j + M_k + M_{(i+j+k \bmod n)} - M_{(i+j \bmod n)} \\ - M_{(i+k \bmod n)} - M_{(j+k \bmod n)}. \end{aligned} \tag{A19}$$

By partially solving the consistent conditions (27)–(29) on $\{h_a^{\text{sc}};m\}$, we obtain a finite set of conditions (A15) and (A18). They are the consistent conditions to be satisfied by the pattern of zeros $\{M_k;p;m\}$, a sequence of integers and half-integers. For instance, the simplest Z_n parafermion states[17,35] correspond to the pattern of zeros $\{M_k=0;p=1;m=2p=2\}$ by choosing non-negative integer $k_m=0$.

APPENDIX B: CONSISTENT CONDITIONS ON THE COMMUTATION FACTOR μ_{AB}

To introduce some useful notations, let us write the OPE between two generic operators $A(z)$ and $B(w)$ as the following:[26]

$$\begin{aligned} A(z)B(w) = \frac{1}{(z-w)^{\alpha_{AB}}}([AB]_{\alpha_{AB}}(w) + (z-w)[AB]_{\alpha_{AB}-1}(w) \\ + (z-w)^2[AB]_{\alpha_{AB}-2}(w) + \cdots), \end{aligned} \tag{B1}$$

where

$$\alpha_{AB} = h_A + h_B - h_{[AB]_{\alpha_{AB}}}, \tag{B2}$$

and h_A ia the scaling dimension of operator A. A and B satisfy the following commutation relation:

$$(z-w)^{\alpha_{AB}}A(z)B(w) = \mu_{AB}(w-z)^{\alpha_{AB}}B(w)A(z). \tag{B3}$$

Let us derive some conditions on μ_{AB} from the associativity of the vertex algebra. By exchanging A and B twice we have

$$\mu_{AB}\mu_{BA} = 1, \tag{B4}$$

which immediately leads to

$$\mu_{AA} = 1. \tag{B5}$$

$\mu_{AA} \neq -1$ because the leading term in the OPE of two A fields would vanish otherwise.

Let

$$B(z)C(w) = \frac{D(w)}{(z-w)^{\alpha_{BC}}} + \cdots. \tag{B6}$$

To exchange A with B and then with C is equivalent to exchange A with $D=[BC]_{\alpha_{BC}}$, so we have

$$\mu_{AB}(-1)^{\alpha_{AB}}\mu_{AC}(-1)^{\alpha_{AC}} = \mu_{AD}(-1)^{\alpha_{AD}} \Rightarrow \mu_{AD} = \mu_{AB}\mu_{AC}r_{ABC}, \tag{B7}$$

in which

$$r_{ABC} \equiv (-1)^{\alpha_{AB}+\alpha_{AC}-\alpha_{AD}} = \pm 1, \tag{B8}$$

and

$$\Delta_3(A,B,C) \equiv \alpha_{AB}+\alpha_{AC}-\alpha_{AD} = h_A^{\text{sc}}+h_B^{\text{sc}}+h_C^{\text{sc}}+h^{\text{sc}}_{[AD]_{\alpha_{AD}}}-h^{\text{sc}}_{[AB]_{\alpha_{AB}}}-h^{\text{sc}}_{[AC]_{\alpha_{AC}}} - h^{\text{sc}}_{[BC]_{\alpha_{BC}}} \in \mathbb{N}. \tag{B9}$$

In a vertex algebra the identity operator 1 (e.g., ψ_0 in a Z_n simple-current vertex algebra) is a zero-scaling-dimension operator with the following OPEs:

$$[1,A]_{-j} = A\delta_{j,0}, \quad [A,1]_{-j} = \frac{1}{j!}\partial^j A, \tag{B10}$$

and $\alpha_{1,A}=\alpha_{A,1}=0$, and $\mu_{1,A}=\mu_{A,1}=1$ for any operator A in the vertex algebra. $\partial^j 1=0$, $j\geqslant 1$ is understood as usual.

One thing needs to be pointed out here: the derivative of an operator A (let us suppose that A is not the identity operator: $A\neq 1$), i.e., ∂A could be zero or not, depending on the definition of this operator A. For example, simple current ψ_2 in a Z_4 Gaffnian vertex algebra obeys $\partial\psi_2=0$. However, this simple current ψ_2 is not the identity operator $1=\psi_0$ since it has nontrivial commutations factors $\mu_{2,1}=\mu_{2,3}=-1\neq 1$.

APPENDIX C: DETERMINE THE QUASIPARTICLE COMMUTATION FACTOR $\mu_{\gamma,a}$ FROM THE QUASIPARTICLE PATTERN OF ZEROS $\{k^{\text{sc}}_{\gamma;a}\}$

The quasiparticle commutation factors $\mu_{\gamma,a}$ are not fully independent of the patter-of-zero data $\{k^{\text{sc}}_{\gamma;a};Q_\gamma\}$. In this section, we try to determine $\mu_{\gamma+b,a}$ from $k^{\text{sc}}_{\gamma;a}$. Note that

$$C_{\gamma+b,a} = \mu_{\gamma+b,a}C_{a,\gamma+b}. \tag{C1}$$

By choosing $A=\sigma_{\gamma+a}$, $B=\psi_b$, $C=\psi_c$, and $D=\psi_{b+c}$ we see from Eq. (B7) that

$$\mu_{\gamma+a,b+c} = \mu_{\gamma+a,b}\mu_{\gamma+a,c}e^{i\pi(\alpha_{\gamma+a,b}+\alpha_{\gamma+a,c}-\alpha_{\gamma+a,b+c})}, \tag{C2}$$

where we have defined

$$\alpha_{\gamma+a,b} \equiv h^{\text{sc}}_{\gamma+a}+h^{\text{sc}}_b-h^{\text{sc}}_{\gamma+a+b}. \tag{C3}$$

Repeatedly using Eq. (C2) we immediately have

$$\mu_{\gamma+a,b} = \mu_{\gamma+a,b-1}\mu_{\gamma+a,1}e^{i\pi(\alpha_{\gamma+a,1}+\alpha_{\gamma+a,b-1}-\alpha_{\gamma+a,b})} = \mu_{\gamma+a,b-2}\mu^2_{\gamma+a,1}e^{i\pi(2\alpha_{\gamma+a,1}+\alpha_{\gamma+1,b-2}-\alpha_{\gamma+a,b})} = \cdots = \mu^b_{\gamma+a,1}e^{i\pi(b\alpha_{\gamma+a,1}-\alpha_{\gamma+a,b})}. \tag{C4}$$

By requiring that $\mu_{\gamma+a,n}=1$ since ψ_n is the identity operator, we have

$$\mu_{\gamma+a,1} = e^{-i\pi\alpha_{\gamma+a,1}+2\pi i[(\kappa_{\gamma+a})/n]}, \tag{C5}$$

where $\kappa_{\gamma+a}$ is an Z_n integer. As a result we can obtain all commutation factors,

$$\mu_{\gamma+a,b} = e^{-i\pi\alpha_{\gamma+a,b}+2\pi i[(b\kappa_{\gamma+a})/n]}. \tag{C6}$$

Due to the consistency condition (B4) we also have

$$\mu_{b,\gamma+a} = e^{i\pi\alpha_{\gamma+a,b}-2\pi i[(b\kappa_{\gamma+a})/n]} \tag{C7}$$

Now we implement Eq. (B7) again with $A=\psi_b$, $B=\sigma_{\gamma+a}$, $C=\psi_c$, and $D=\sigma_{\gamma+a+c}$ to see whether there are any new consistency conditions for quasiparticle scaling dimensions $h^{\text{sc}}_{\gamma+a}$,

$$1 = \mu_{b,\gamma+a}\mu_{b,c}\mu_{\gamma+a+c,b}e^{i\pi(\alpha_{\gamma+a,b}+\alpha_{b,c}-\alpha_{b,\gamma+a+c})} = e^{2\pi i(\alpha_{\gamma+a,b}+\alpha_{b,c}-\alpha_{\gamma+a+c,b})}\mu_{b,c}e^{-i\pi\alpha_{b,c}+2\pi i\{[b(\kappa_{\gamma+a+c}-\kappa_{\gamma+a})]/n\}} = e^{-i\pi bc\alpha_{1,1}+2\pi i\{[b(\kappa_{\gamma+a+c}-\kappa_{\gamma+a})]/n\}} = e^{2\pi ib[(\kappa_{\gamma+a+c}-pc-\kappa_{\gamma+a})/n]}, \tag{C8}$$

where we used Eq. (E8), $q\equiv n\alpha_{1,1}/2\in\mathbb{Z}$, and that $\Delta_3(\gamma+a,b,c)=\alpha_{\gamma+a,b}+\alpha_{b,c}-\alpha_{\gamma+a+c,b}\in\mathbb{N}$. Equation (B7) with $A=\sigma_{\gamma+a}$, $B=\psi_b$, $C=\psi_c$, $D=\psi_{b+c}$ does not produce any new conditions. We can see that all the consistency conditions on $\{\mu_{\gamma+a,b}\}$, i.e., Eqs. (B4) and (B7) can be guaranteed by choosing Eqs. (C6) and (C7) and the integer $\kappa_{\gamma+a}$ as

$$\kappa_{\gamma+a} = \kappa_\gamma + qa \bmod n. \tag{C9}$$

We find that $\mu_{\gamma+a,b}$ and $\mu_{b,\gamma+a}$ can almost be determined from $k^{\text{sc}}_{\gamma;a}$

$$\mu_{\gamma+a,b} = e^{-i\pi\alpha_{\gamma+a,b}+2\pi i\{[b(\kappa_\gamma+aq)]/n\}},$$

$$\mu_{b,\gamma+a} = e^{i\pi\alpha_{\gamma+a,b}-2\pi i\{[b(\kappa_\gamma+aq)]/n\}},$$

$$q = n\alpha_{1,1}/2 \in \mathbb{Z}. \tag{C10}$$

However, we need to supply a Z_n integer κ_γ to fully fix $\mu_{\gamma+a,b}$ and $\mu_{b,\gamma+a}$ from $k^{\text{sc}}_{\gamma;a}$. (Note that $\alpha_{\gamma+a,b}=h^{\text{sc}}_b-\Sigma_{i=a}^{a+b}k^{\text{sc}}_{\gamma;i}$.)

APPENDIX D: GENERALIZED JACOBI IDENTITY

1. GJI's of an associative vertex algebra

In the above, we only considered the associativity of the vertex algebra through the commutation factor μ_{AB}. Although some new conditions on pattern of zeros and some relations between the quasiparticle scaling dimensions are obtained, the associativity of the algebra is not fully utilized. To fully use the associativity condition of the vertex algebra, we need to derive the generalized Jacobi Identity.

Choose $f(z,w)$ in the following relation:

$$\oint_{|z|>|w|} dz \oint_0 dw f(z,w) - \oint_{|w|>|z|} dw \oint_0 dz f(z,w) = \oint_0 dw \oint_w dz f(z,w) \tag{D1}$$

to be the operator function

$$f(z,w) = A(z)B(w)C(0)(z-w)^{\gamma_{AB}} w^{\gamma_{BC}} z^{\gamma_{AC}}, \tag{D2}$$

with $\alpha_{AB}-\gamma_{AB}$, $\alpha_{AC}-\gamma_{AC}$, $\alpha_{BC}-\gamma_{BC}\in\mathbb{Z}$, we obtain the generalized Jacobi identity (GJI),

$$\sum_{j=0}^{\alpha_{BC}-\gamma_{BC}-1} (-1)^j \binom{\gamma_{AB}}{j} [A[BC]_{\gamma_{BC}+j+1}]_{\gamma_{AB}+\gamma_{AC}+1-j} - \mu_{AB}(-1)^{\alpha_{AB}-\gamma_{AB}} \sum_{j=0}^{\alpha_{AC}-\gamma_{AC}-1} (-1)^j \binom{\gamma_{AB}}{j} [B[AC]_{\gamma_{AC}+j+1}]_{\gamma_{AB}+\gamma_{BC}+1-j}$$
$$= \sum_{j=0}^{\alpha_{AB}-\gamma_{AB}-1} \binom{\gamma_{AC}}{j} [[AB]_{\gamma_{AB}+j+1}C]_{\gamma_{BC}+\gamma_{AC}+1-j}, \tag{D3}$$

where $\binom{n}{m}$ is the binomial function.

When we choose $\gamma_{AB}-\alpha_{AB}=\gamma_{AC}-\alpha_{AC}=\gamma_{BC}-\alpha_{BC}=0$, Eq. (D2) is a regular function with the asymptotic behavior

$$\lim_{z\to0}\lim_{w\to0} f(z,w) = \lim_{z\to0} z^{\alpha_{AB}+\alpha_{AC}} A(z)D(0) = \lim_{z\to0} z^{\alpha_{AB}+\alpha_{AC}-\alpha_{AD}} [AD]_{\alpha_{AD}}(0). \tag{D4}$$

Since $f(z,w))$ is an analytic function of both z and w, $z^{\alpha_{AB}+\alpha_{AC}-\alpha_{AD}}$ should still be an analytic function of z. Thus $\alpha_{AB}+\alpha_{AC}-\alpha_{AD}$ should be a non-negative integer, allowing us to obtain the consistency condition (B9).

For clarity we introduce three integers n_{AB}, n_{AC}, and n_{BC} as

$$\gamma_{AB} = \alpha_{AB} - 1 - n_{AB}, \quad \gamma_{AC} = \alpha_{AC} - 1 - n_{AC}, \quad \gamma_{BC} = \alpha_{BC} - 1 - n_{BC}, \tag{D5}$$

and the GJI (D3) can be rewritten as

$$(-1)^{n_{BC}} \sum_{j=0}^{n_{BC}} (-1)^j \binom{\alpha_{AB}-1-n_{AB}}{n_{BC}-j} [A[BC]_{\alpha_{BC}-j}]_{\alpha_{AB}+\alpha_{AC}-1-(n_{AB}+n_{AC}+n_{BC})+j} + \mu_{AB}(-1)^{n_{AB}+n_{AC}} \sum_{j=0}^{n_{AC}} (-1)^j \binom{\alpha_{AB}-1-n_{AB}}{n_{AC}-j}$$
$$\times [B[AC]_{\alpha_{AC}-j}]_{\alpha_{AB}+\alpha_{BC}-1-(n_{AB}+n_{AC}+n_{BC})+j} = \sum_{j=0}^{n_{AB}} \binom{\alpha_{AC}-n_{AC}-1}{n_{AB}-j} [[AB]_{\alpha_{AB}-j}C]_{\alpha_{BC}+\alpha_{AC}-1-(n_{AB}+n_{AC}+n_{BC})+j}. \tag{D6}$$

The GJI (D3) and (D6) is the associativity condition of a vertex algebra. It generalizes the usual Jacobi identity of a Lie algebra to the case of an infinite-dimensional Lie algebra (the vertex algebra here), with the usual Lie bracket (the commutator) defined by OPE in Eq. (B1) We say that the theory is associative up to a certain order if all the GJI's are satisfied up to this order in OPE. Applying the GJI, more conditions on the patterns of zeros can be found. More importantly, those conditions are likely to be the necessary and the sufficient conditions.

For example, by choosing C in GJI (D6) to be the identity operator 1 [note that we have $\Delta_3(A,B,1)=0$], $n_{AB}=k\geq0$, $n_{BC}=-1$, and $n_{AC}=0$ and making use of OPE (B10), we immediately reach the following relation:

$$\mu_{AB}[BA]_{\alpha_{AB}-k} = \sum_{j=0}^{k} \frac{(-1)^j}{(k-j)!} \partial^{k-j} [AB]_{\alpha_{AB}-j}. \tag{D7}$$

This allows us to obtain the OPE of $[B,A]$ to the same order with the OPE of $[A,B]$ to a certain order in hand. For example, we have

$$[\psi_i T]_1 = \partial[T\psi_i]_2 - [T\psi_i]_1 = (h_i^{\mathrm{sc}}-1)\partial\psi_i$$

since $\mu_{T,\psi_i}=1$ and $\alpha_{T,\psi_i}=2$. As a special case of Eq. (D7) we have

$$[AA]_{\alpha_{AA}-2k-1} = \frac{1}{2}\sum_{j=0}^{2k} \frac{(-1)^j}{(2k+1-j)!} \partial^{2k+1-j} [AA]_{\alpha_{AA}-j}. \tag{D8}$$

This relation is actually an example, showing how we "derive" (or more precisely, obtain the consistent conditions of) higher order OPE's from the known OPE's up to a certain order based on GJI's.

2. "Useful" GJI's of a vertex algebra up to a certain order

In practice we need to extract the consistent conditions of a vertex algebra from a set of "useful" GJI's concerning only the OPE's up to a certain order. The OPE's up to this order are determined already except for some structure constants (usually complex numbers). Other GJI's involving higher order OPE's do not serve as constraints to the vertex algebra up to this order since we can always introduce new operators into this infinite-dimensional Lie algebra in higher order OPE's. For example, a generic Z_n vertex algebra is defined by OPE's between currents $\{\psi_a\}$ up to leading order with $[\psi_i,\psi_j]_{\alpha_{i,j}}=C_{i,j}\psi_{i+j}$.

Let us now consider the GJI (D6) of three operators (A,B,C), with the corresponding vertex algebra defined up to (N_{AB},N_{BC},N_{AC}) order, i.e., $[AB]_{\alpha_{AB}-i}$ is known up to structure constants for all $0\leqslant i\leqslant N_{AB}$ in the OPE (B1) of operators A and B. For example, in a special Z_n simple-current vertex algebra defined by OPE's (67)–(70), with $(A=\psi_i,B=\psi_j)$ we have $N_{AB}=0$ if $i+j\neq 0 \bmod n$ or $N_{AB}=2$ if $i+j=0 \bmod n$. Let us further assume the following relation:

$$\alpha_{[AB]_{\alpha_{AB}-j},C}=\alpha_{[AB]_{\alpha_{AB}},C}+j, \quad \text{if } [AB]_{\alpha_{AB}-j}\neq 0 \tag{D9}$$

is satisfied by any operators (A,B,C) of this vertex algebra. It is straightforward to verify that Z_n simple-current vertex algebra indeed obeys the above relation: e.g., $(A=\psi_i,B=\psi_{n-i},C=\psi_j)$ we have $1=[AB]_{\alpha_{AB}}$, $T=[AB]_{\alpha_{AB}-2}$ and $\alpha_{T,\psi_j}=2$, $\alpha_{1,\psi_j}=0$. Defining the following quantity:

$$\begin{aligned}N_{ABC}(n_{AB},n_{BC},n_{AC}) \equiv \min\{&N_{[AB]_{\alpha_{AB}-j},C},0\leqslant j\\ &\leqslant N_{AB};N_{B,[AC]_{\alpha_{AC}-j}},0\leqslant j\\ &\leqslant N_{AC};N_{A,[BC]_{\alpha_{BC}-j}},0\leqslant j\leqslant N_{BC}\},\end{aligned} \tag{D10}$$

then we can obtain all the "useful" consistent conditions of the vertex algebra from the GJI (D6), by choosing

$$\begin{aligned}&n_{AB}\leqslant N_{AB},\quad n_{BC}\leqslant N_{BC},\quad n_{AC}\leqslant N_{AC},\\ &\Delta_3(A,B,C)-1\leqslant n_{AB}+n_{BC}+n_{AC}\\ &\qquad\leqslant \Delta_3(A,B,C)-1+N_{ABC}(n_{AB},n_{BC},n_{AC}).\end{aligned} \tag{D11}$$

Any other choice with larger (n_{AB},n_{AC},n_{BC}) will involve higher order OPE's. Generally speaking, the set of useful GJI's satisfying Eq. (D11) will be translated into a set of nonlinear equations of structure constants. [Here "structure constants" have a broader meaning than usual, e.g., $2h_i^{\text{sc}}/c$ in Eq. (68) and h_i^{sc} in Eq. (69) should also be considered as structure constants.] Some of these equations become consistent conditions of this vertex algebra, while others help define this vertex algebra, e.g., by determining the structure constants $\{C_{a,b},C_{a,\gamma+b}\}$ and central charge c of a Z_n simple-current vertex algebra, as is shown in Secs. V and VII.

APPENDIX E: ASSOCIATIVITY OF Z_n VERTEX ALGEBRA AND NEW CONDITIONS ON h_a^{sc} AND C_{ab}

In this section, we apply the consistency conditions of commutation factor discussed in Appendixes B and GJI discussed in Appendix D to a Z_n vertex algebra. This allows us to derive additional conditions on the scaling dimension h_a^{sc} from the associativity of Z_n vertex algebra.

As mentioned earlier, a generic FQH wave function can be expressed as a correlation function of an associative vertex algebra obeying the following OPE:

$$\psi_a(z)\psi_b(w)=\frac{C_{a,b}\psi_{a+b}(w)+O(z-w)}{(z-w)^{\alpha_{a,b}}}, \tag{E1}$$

where we have $\alpha_{a,b}\equiv h_a^{\text{sc}}+h_b^{\text{sc}}-h_{a+b \bmod n}^{\text{sc}}$. This guarantees the quasi-Abelian fusion rule $\psi_a\psi_b\sim\psi_{a+b}$ [see Eq. (34)]. Moreover, we choose the normalization of simple currents ψ_a to be Eq. (44).

1. New conditions from the commutation factors

If we use the radial order (38) to calculate correlation function, then the continuity of the correlation function requires that

$$(z-w)^{\alpha_{V_aV_b}}V_a(z)V_b(w)=\mu_{V_a,V_b}(w-z)^{\alpha_{V_aV_b}}V_b(w)V_a(z). \tag{E2}$$

Since the operators $e^{ia\phi(z)/\sqrt{\nu}}$ in the Gaussian model satisfy

$$\begin{aligned}&(z-w)^{(a^2/2\nu)+(b^2/2\nu)-[(a+b)^2/2\nu]}e^{ia\phi(z)/\sqrt{\nu}}e^{ib\phi(w)/\sqrt{\nu}}\\ &\quad=(w-z)^{(a^2/2\nu)+(b^2/2\nu)-[(a+b)^2/2\nu]}e^{ib\phi(w)/\sqrt{\nu}}e^{ia\phi(z)/\sqrt{\nu}},\end{aligned} \tag{E3}$$

the simple current operator satisfy the following commutation relation:

$$(z-w)^{\alpha_{ab}}\psi_a(z)\psi_b(w)=\mu_{ab}(w-z)^{\alpha_{ab}}\psi_b(w)\psi_a(z), \tag{E4}$$

where

$$\alpha_{ab}=h_a^{\text{sc}}+h_b^{\text{sc}}-h_{a+b}^{\text{sc}}. \tag{E5}$$

We stress that relation (E4) is required by the continuity of the correlation function of the electron operators.

The commutation factors μ_{ij} satisfy some consistency relations, which is discussed in Appendix B under a more general setting. Conditions (B4), (B5), and (B7) are the conditions on the commutation factors μ_{ij} that were obtained from the associativity of the vertex algebra. From the n-cluster condition $\psi^n=\psi_n=1$, we also have

$$\mu_{ij}=1 \quad \text{if } i=0 \bmod n \text{ or } j=0 \bmod n \tag{E6}$$

due to the definition of the identity operator $\psi_0=1$ shown in Appendix B.

Those conditions, (B4), (B5), (B7), and (E6), can be expressed as the extra condition on the scaling dimensions h_a^{sc}. We note that according to Eqs. (B7) and (B8),

$$\begin{aligned}\mu_{ij}&=\mu_{i,1}\mu_{i,j-1}(-1)^{\alpha_{i,1}+\alpha_{i,j-1}-\alpha_{i,j}}\\ &=\mu_{i,1}^2\mu_{i,j-2}(-1)^{2\alpha_{i,1}+\alpha_{i,j-2}-\alpha_{i,j}}\\ &=\cdots=\mu_{i,1}^j(-1)^{j\alpha_{i,1}-\alpha_{ij}}.\end{aligned} \tag{E7}$$

A similar manipulation leads to $\mu_{1,i}=\mu_{1,1}^i(-1)^{i\alpha_{1,1}-\alpha_{1,i}}$, and we can write the commutation factor in a symmetric way

$$\mu_{ij}=(-1)^{ij\alpha_{1,1}-\alpha_{i,j}}=\pm 1. \tag{E8}$$

We see that μ_{ij} can be expressed in terms of h_i^{sc}. Equation (E8) also implies that

$$ij\alpha_{1,1}-\alpha_{i,j}=\text{integer}, \tag{E9}$$

which is actually guaranteed by Eq. (29). Condition (E6) becomes

TABLE IX. The parity of M_k with different parity of n for a special Z_n simple-current vertex algebra with OPE (68), according to constraint (E17). The parafermion scaling dimension $\{h_a^{\text{sc}}\}$ is given by $h_a^{\text{sc}}=pa(n-a)/n-M_a$, with p being a non-negative integer.

$n=\text{odd}$	$M_k \in 2\mathbb{Z}$	
$n=\text{even}$	$M_{2\mathbb{Z}+1} \in \mathbb{Z}$	$M_{2\mathbb{Z}} \in 2\mathbb{Z}$

$$ij\alpha_{1,1} - \alpha_{i,j} = \text{even} \quad \text{if } i = 0 \bmod n \text{ or } j = 0 \bmod n. \tag{E10}$$

Now we use $\{M_k, k=1,3,4,\ldots,[\frac{n}{2}];p\}$ to describe h_a^{sc} (see Appendix A 2). So the consistent conditions on $\{h_a^{\text{sc}}\}$ can be translated into the consistent conditions on $\{M_k;p\}$. We note that

$$\begin{aligned}\alpha_{i,j} &\equiv \alpha_{\psi_i,\psi_j}\\ &= h_i^{\text{sc}} + h_j^{\text{sc}} - h_{i+j}^{\text{sc}}\\ &= p\frac{2ij}{n} - M_i - M_j + M_{i+j \bmod n} - 2p(i+j-n)\theta(i+j-n)\end{aligned} \tag{E11}$$

for $1 \leqslant i,j \leqslant n-1$.

By choosing $j=n$ in Eq. (E10) we have

$$n\alpha_{1,1} = 2p = \text{even}. \tag{E12}$$

As a result

$$p \in \mathbb{N}. \tag{E13}$$

Besides, $\mu_{i,i}=1$ becomes another constraint

$$i^2\alpha_{1,1} - \alpha_{i,i} = \text{even} \quad \forall\, i = 1,2,\ldots,n-1. \tag{E14}$$

What is more from OPE (68) of a special Z_n simple current vertex algebra and the definition of commutation factor (E4) we immediately have

$$\mu_{a,b} = 1 \quad \text{if } a+b = 0 \bmod n, \tag{E15}$$

which becomes an extra constraint

$$i(n-i)\alpha_{1,1} - \alpha_{i,n-i} = \text{even} \quad \forall\, i = 1,2,\ldots,n-1. \tag{E16}$$

If we require OPE (68) to be satisfied, combining Eqs. (E11), (E14), and (E16) we find

$$2M_i - M_{2i \bmod n} = 2M_i = \text{even}. \tag{E17}$$

This determines the parity of $\{M_k\}$ as summarized in Table IX. Notice that we always have $M_k \in \mathbb{Z}$ for such a special Z_n vertex algebra.

However, constraint (E16) is too strong and is not necessary (this is why we use special as a description here). For example, a four-cluster state called Gaffnian explicitly violates it since we have $\mu_{1,3}=-1 \neq 1$ for a Gaffnian vertex algebra. To remove constraint (E16), we need to modify the normalization $C_{a,-a}=1$ in OPE (40) to for $a \leqslant n/2 \bmod n$

TABLE X. The parity of M_k with different parity of n for a Z_n simple current vertex algebra with OPE (71), according to constraint (E20). The parafermion scaling dimension $\{h_a^{\text{sc}}\}$ is given by $h_a^{\text{sc}}=pa(n-a)/n-M_a$, with p being a non-negative integer.

$n=\text{odd}$	$M_k \in 2\mathbb{Z}$	
$n=2 \bmod 4$	$M_{2\mathbb{Z}+1} \in \mathbb{Z}$	$M_{2\mathbb{Z}} \in 2\mathbb{Z}$
$n=0 \bmod 4$	$2M_i - M_{2i \bmod n} \in 2\mathbb{Z}$	$M_{4\mathbb{Z}} \in 2\mathbb{Z}$

$$\psi_a(z)\psi_{-a}(w) = \frac{1}{(z-w)^{2h_a^{\text{sc}}}} + \cdots, \tag{E18}$$

and for $a > n/2 \bmod n$

$$\psi_a(z)\psi_{-a}(w) = \frac{\mu_{a,-a}}{(z-w)^{2h_a^{\text{sc}}}} + \cdots. \tag{E19}$$

If we adopt the more general OPEs (E18) and (E19), the corresponding consistent conditions on $\{M_k\}$ from Eqs. (E11) and (E14) become

$$2M_i - M_{2i \bmod n} = \text{even}. \tag{E20}$$

This leads to some conclusions on the parity of $\{M_k\}$ as summarized in Table X. Generally we do not have $M_k \in \mathbb{Z}$ for such a generic Z_n vertex algebra.

To summarize, Eqs. (E12) and (E14) are the extra conditions from commutation factors for a generic Z_n vertex algebra. They become Eqs. (E13) and (E20) when translated into consistent conditions of $\{M_k;p;m\}$, as a supplement to conditions (A15)–(A18) in Appendix A 2.

2. New conditions from GJI

As shown in Appendix D, all GJI's must be satisfied for the associativity of the vertex algebra. With the OPE (E1), we have $N_{a,b} \equiv N_{\psi_a,\psi_b}=0$ and the useful GJI's are very limited. We list the consistent conditions from useful GJI's of this generic Z_n vertex algebra below. Then we summarize the new consistent conditions on $\{h_a^{\text{sc}}\}$ and $\{c\}$.

a. List of useful GJI's ($A=\psi_a$, $B=\psi_b$, $C=\psi_c$)

Using the notations in Appendix D, here we have $N_{AB}=N_{BC}=N_{AC}=0$.

If $\Delta_3(a,b,c)=0$, all the three useful GJI's satisfying Eq. (D11) are

$$n_{AB} = -1, \quad n_{BC} = 0,$$

$$n_{AC} = 0 \Rightarrow (C_{b,c}C_{a,b+c} - \mu_{a,b}C_{a,c}C_{b,a+c})\psi_{a+b+c} = 0,$$

$$n_{AB} = 0, \quad n_{BC} = -1,$$

$$n_{AC} = 0 \Rightarrow (\mu_{a,b}C_{a,c}C_{b,a+c} - C_{a,b}C_{a+b,c})\psi_{a+b+c} = 0,$$

$$n_{AB} = 0, \quad n_{BC} = 0,$$

$$n_{AC}=-1 \Rightarrow (C_{b,c}C_{a,b+c}-C_{a,b}C_{a+b,c})\psi_{a+b+c}=0.$$

If $\Delta_3(a,b,c)=1$, the only 1 useful GJI satisfying Eq. (D11) is

$$n_{AB}=0, \quad n_{BC}=0,$$

$$n_{AC}=0 \Rightarrow (C_{a,b}C_{a+b,c}-C_{b,c}C_{a,b+c}-\mu_{a,b}C_{a,c}C_{b,a+c})\psi_{a+b+c} =0.$$

For $\Delta_3(a,b,c)\geqslant 2$ there are no useful GJI's satisfying Eq. (D11) and thus no new conditions on $C_{i,j}$.

b. Summary of new consistent conditions from GJI

As shown in Appendix E 2 a, for $\Delta_3(a,b,c) \equiv \Delta_3(\psi_a,\psi_b,\psi_c)=0$ the extra consistent conditions are

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}=\mu_{a,b}C_{a,c}C_{b,a+c}. \tag{E21}$$

For $\Delta_3(a,b,c)=1$ the corresponding consistent condition is

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}+\mu_{a,b}C_{a,c}C_{b,a+c}. \tag{E22}$$

For $\Delta_3(a,b,c)\geqslant 2$ there are no useful GJI's and no extra consistent conditions.

The above conditions should be satisfied no matter what (a,b,c) are. Now let us further specify (a,b,c) and use the normalization (44) of structure constants to obtain new conditions.

If $a+b,a+c,b+c\neq 0 \bmod n$. For $\Delta_3(a,b,c)=0$ we have

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}=\mu_{a,b}C_{a,c}C_{b,a+c}. \tag{E23}$$

For $\Delta_3(a,b,c)=1$ we have

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}+\mu_{a,b}C_{a,c}C_{b,a+c}. \tag{E24}$$

If $a\pm b\neq 0 \bmod n$. For $\Delta_3(a,b,-b)=0$ we have

$$C_{a,b}C_{a+b,-b}=C_{b,-b}=\mu_{a,b}C_{a,-b}C_{b,a-b}. \tag{E25}$$

For $\Delta_3(a,b,-b)=1$ we have

$$C_{a,b}C_{a+b,-b}=C_{b,-b}+\mu_{a,b}C_{a,-b}C_{b,a-b}. \tag{E26}$$

If $a\neq n/2 \bmod n$. For $\Delta_3(a,a,-a)=0$ we have

$$\mu_{a,-a}=C_{a,-a}=C_{a,a}C_{2a,-a}=1. \tag{E27}$$

For $\Delta_3(a,a,-a)=1$ we have

$$\mu_{a,-a}=\mu_{-a,a}=-1, \tag{E28}$$

$$C_{a,a}C_{2a,-a}=2C_{a,-a}. \tag{E29}$$

If n=even, we require $\Delta_3(\frac{n}{2},\frac{n}{2},\frac{n}{2})\neq 1$ since otherwise

$$\psi_{n/2}\equiv 0 \tag{E30}$$

must be required to satisfy GJI's.

Among the above consistent conditions, some are just conditions on the structure constants $\{C_{a,b}\}$, while others serve as the new conditions on the pattern of zeros $\{h_a^{\rm sc}\}$ or $\{S_a\}$. As a summary, the extra consistent conditions for the pattern of zeros from GJI's are

$$\mu_{a,-a}=1 \quad \text{if } \Delta_3(a,a,-a)=0,$$

$$\mu_{a,-a}=-1 \quad \text{if } \Delta_3(a,a,-a)=1,$$

$$\Delta_3\left(\frac{n}{2},\frac{n}{2},\frac{n}{2}\right)=4h_{n/2}^{\rm sc}\neq 1, \quad n=\text{even}. \tag{E31}$$

where we need Eq. (E8) to relate commutation factor $\mu_{a,b}$ with the pattern of zeros. Note that the first two conditions in the above can be rewritten as $a^2\alpha_{1,1}+\alpha_{a,-a}$=even if $\Delta_3(a,a,-a)=0$ and $a^2\alpha_{1,1}+\alpha_{a,-a}$=odd if $\Delta_3(a,a,-a)=1$. Since $a^2\alpha_{1,1}-\alpha_{a,a}$=even and $\Delta_3(a,a,-a)=\alpha_{a,a}+\alpha_{a,-a}$, the two conditions are always satisfied.

Obviously these extra conditions, based on the most generic OPE (E1) of a Z_n vertex algebra, are not enough to determine the structure constants $\{C_{a,b}\}$ of this vertex algebra. In order to have more consistent conditions and to determine the structure constants, we need to specify higher order terms in the OPE (E1). This is done through defining Z_n simple current vertex algebra in Sec. V, essentially by introducing the energy momentum tensor T and Virasoro algebra into the vertex algebra. The corresponding extra consistent conditions are summarized in Secs. V B and VI D. We can obtain even more extra conditions from GJI's when we fix the subleading term of OPE's between simple currents, as shown in Appendix F.

APPENDIX F: SUBLEADING TERMS IN OPE OF A Z_n SIMPLE-CURRENT VERTEX ALGEBRA AND MORE CONSISTENT CONDITIONS

1. "Deriving" subleading terms in OPE from GJI's

In this section we show how to "derive" the subleading term in OPE (67) of a Z_n simple-current vertex algebra as an example. First we notice that the subleading term $[\psi_a\psi_b]_{\alpha_{a,b}-1}$ should have a scaling dimension of $h_{a+b}^{\rm sc}+1$, thus we propose the following conclusion:

$$[\psi_a\psi_b]_{\alpha_{a,b}-1}=C_{a,b}d_{a,b}\partial\psi_{a+b}. \tag{F1}$$

Then we can use GJI's to determine the expression of $d_{a,b}$ in terms of scaling dimensions $\{h_i^{\rm sc}\}$.

First we choose $(A=T, B=\psi_a, C=1)$ and ($n_{AB}=-1 n_{BC}=1$, $n_{AC}=-1$) in GJI (D6). Since $\Delta_3(T,\psi_a,1)=0$ and $[\psi_a,1]_1 =\partial\psi_a$ we have the following consistent conditions from this GJI:

$$[T\partial\psi_a]_3=2[T\psi_a]_2=2h_a^{\rm sc}\psi_a. \tag{F2}$$

Then we choose $(A=T, B=\psi_a, C=\psi_b)$, $a+b\neq 0 \bmod n$ and ($n_{AB}=0$, $n_{BC}=1$, $n_{AC}=0$) in GJI (D6). It's easy to verify that $\Delta_3(T,\psi_a,\psi_b)=2$ since we have $\alpha_{T,\psi_i}=2$, $\forall i$. Plugging in Eq. (F1) and this GJI yields a consistent condition on $d_{a,b}$,

$$2d_{a,b}h_{a+b}^{\rm sc}=(h_{a+b}^{\rm sc}+h_a^{\rm sc}-h_b^{\rm sc}). \tag{F3}$$

Thus we conclude that as long as $h_{a+b}^{\rm sc}\neq 0$ (this should hold in most cases except for "strange" examples like Gaffnian) we have

$$d_{a,b}=\frac{1}{2}\left(1+\frac{h_a^{\mathrm{sc}}-h_b^{\mathrm{sc}}}{h_{a+b}^{\mathrm{sc}}}\right). \tag{F4}$$

2. More consistent conditions due to subleading terms in OPE

Now with the subleading terms we have more useful GJI's and therefore more consistent conditions on the data $\{n;m;h_a^{\mathrm{sc}};c\}$ characterizing a Z_n simple current vertex algebra. In this section we shall show the extra consistent conditions accompanied with the introduction of the subleading term (F1) and (F4) in OPE of $[\psi_a,\psi_b]$.

It turns out that there are many more useful GJI's considering the subleading order OPE (F1) with Eq. (F4). In many cases the new consistent conditions are extremely complicated, so we will only show the complete consistent conditions in several cases (which will be utilized in Sec. VII for some examples).

a. $\{A,B,C\}=\{\psi_a,\psi_b,\psi_c\}, a+b, b+c, a+c, a+b+c\neq 0 \bmod n$

Right now we have $N_{AB}=N_{BC}=N_{AC}=1>0$ thus there are more useful GJI's and more conditions compared with Sec. V B 1.

For $\Delta_3(a,b,c)=0$ the complete consistent conditions are summarized as

$$C_{a,b}C_{a+b,c}=C_{b,c}C_{a,b+c}=\mu_{a,b}C_{a,c}C_{b,a+c},$$

$$\alpha_{a,b+c}d_{b,c}=\alpha_{b,a+c}d_{a,c}=\alpha_{a,b},$$

$$(d_{a,b+c}+d_{b,a+c}-d_{a+b,c})\partial\psi_{a+b+c}\equiv 0,$$

$$[d_{a,b+c}-d_{a,c}(1-d_{b,a+c})]\partial\psi_{a+b+c}\equiv 0,$$

$$(d_{a,b+c}-d_{a,c}d_{a+c,b})\partial\psi_{a+b+c}\equiv 0. \tag{F5}$$

The above conditions should also be satisfied with respect to any permutations of (a,b,c).

For $\Delta_3(a,b,c)=1$ some of the new consistent conditions are

$$\frac{C_{b,c}C_{a,b+c}}{C_{a,b}C_{a+b,c}}=(1-\alpha_{a,b}+d_{b,c}\alpha_{a,b+c})^{-1},$$

$$\frac{\mu_{a,b}C_{a,c}C_{b,a+c}}{C_{a,b}C_{a+b,c}}=\frac{\alpha_{a,b}-d_{b,c}\alpha_{a,b+c}}{\alpha_{a,b}-d_{b,c}\alpha_{a,b+c}-1},$$

$$(\alpha_{a,b}+d_{a,c}\alpha_{b,a+c})(\alpha_{a,b}-d_{b,c}\alpha_{a,b+c})=1,$$

$$(1-\alpha_{a,b}+d_{b,c}\alpha_{a,b+c})(\alpha_{a,c}-d_{a,b}\alpha_{a+b,c})=1. \tag{F6}$$

We did not show those lengthy consistent conditions with the form of $(\cdots)\partial\psi_{a+b+c}=0$ here.

For $\Delta_3(a,b,c)=2$ the new consistent conditions are

$$(\alpha_{a,b}-d_{b,c}\alpha_{a,b+c})C_{b,c}C_{a,b+c}=\mu_{a,b}C_{a,c}C_{b,a+c}(\alpha_{a,b}-\alpha_{b,a+c}d_{a,c}), \tag{F7}$$

$$(\alpha_{a,b}-d_{b,c}\alpha_{a,b+c}-2)C_{b,c}C_{a,b+c}=-C_{a,b}C_{a+b,c}(\alpha_{a,c}-\alpha_{a+b,c}d_{a,b}), \tag{F8}$$

$$(\alpha_{a,b}-d_{a,c}\alpha_{b,a+c}-2)C_{a,b}C_{a+b,c}=\mu_{a,b}C_{a,c}C_{b,a+c}(\alpha_{a,b}-\alpha_{b,a+c}d_{a,c}-2), \tag{F9}$$

$$\begin{aligned}0\equiv\partial\psi_{a+b+c}\{&C_{a,b}C_{a+b,c}d_{a+b,c}+C_{b,c}C_{a,b+c}[d_{a,b+c}(\alpha_{a,b}+d_{b,c}-1-d_{b,c}\alpha_{a,b+c})-d_{b,c}]\\&+\mu_{a,b}C_{a,c}C_{b,a+c}[d_{b,a+c}(\alpha_{a,b}+d_{a,c}-1-d_{a,c}\alpha_{b,a+c})-d_{a,c}]\},\end{aligned} \tag{F10}$$

$$\begin{aligned}0\equiv\partial\psi_{a+b+c}\{&C_{b,c}C_{a,b+c}d_{a,b+c}+C_{a,b}C_{a+b,c}d_{a+b,c}(-\alpha_{a,c}-d_{a,b}+1+d_{a,b}\alpha_{a+b,c})\\&+\mu_{a,b}C_{a,c}C_{b,a+c}[d_{b,a+c}(\alpha_{a,b}+d_{a,c}-2-d_{a,c}\alpha_{b,a+c})-d_{a,c}]\},\end{aligned} \tag{F11}$$

$$\begin{aligned}0\equiv\partial\psi_{a+b+c}\{&\mu_{a,b}C_{a,c}C_{b,a+c}d_{b,a+c}+C_{b,c}C_{a,b+c}[d_{a,b+c}(\alpha_{a,b}+d_{b,c}-2-d_{b,c}\alpha_{a,b+c})-d_{b,c}]\\&+C_{a,b}C_{a+b,c}d_{a+b,c}(\alpha_{a,c}+d_{a,b}-1-d_{a,b}\alpha_{a+b,c})\}.\end{aligned} \tag{F12}$$

For $\Delta_3(a,b,c)=4$ the new consistent conditions are

$$n_{AB}=1,\quad n_{BC}=1,$$

$$n_{AC}=1\Rightarrow C_{a,b}C_{a+b,c}(2-\alpha_{a,c}+d_{a,b}\alpha_{a+b,c})+C_{b,c}C_{a,b+c}(2-\alpha_{a,b}+d_{b,c}\alpha_{a,b+c})-\mu_{a,b}C_{a,c}C_{b,a+c}(2-\alpha_{a,b}+d_{a,c}\alpha_{b,a+c})=0. \tag{F13}$$

For $\Delta_3(a,b,c)\geqslant 5$ there are no useful GJI's and thus no extra consistent conditions.

b. $\{A,B,C\}=\{\psi_a,\psi_b,\psi_{-a-b}\}, a,b,a+b\neq 0 \bmod n$

For $\Delta_3(a,b,-a-b)=0$ the new consistent conditions are

$$
\begin{aligned}
&C_{a,b}C_{a+b,-a-b}=C_{b,-a-b}C_{a,-a}=\mu_{a,b}C_{a,-a-b}C_{b,-b},\\
&\alpha_{a,b}=\alpha_{a,-a}d_{b,-a-b}=\alpha_{b,-b}d_{a,-a-b},\\
&\alpha_{a,-a-b}=\alpha_{a,-a}d_{-a-b,b}=\alpha_{-a-b,a+b}d_{a,b},\\
&d_{b,-a-b}h^{\mathrm{sc}}_a=d_{a,-a-b}h^{\mathrm{sc}}_b=\alpha_{a,b}/2,\\
&d_{-a-b,b}h^{\mathrm{sc}}_a=d_{a,b}h^{\mathrm{sc}}_{a+b}=\alpha_{a,-a-b}/2,\\
&d_{a,b}h^{\mathrm{sc}}_a=\alpha_{a,-a-b}(h^{\mathrm{sc}}_a+1-h^{\mathrm{sc}}_{a+b})/2,\\
&d_{-a-b,b}h^{\mathrm{sc}}_{a+b}=\alpha_{-a-b,a}(h^{\mathrm{sc}}_{a+b}+1-h^{\mathrm{sc}}_a)/2,\\
&d_{a,-a-b}h^{\mathrm{sc}}_a=\alpha_{a,b}(h^{\mathrm{sc}}_a+1-h^{\mathrm{sc}}_b)/2.
\end{aligned}
\tag{F14}
$$

The above conditions should also be satisfied with $a\leftrightarrow b$ exchange.

For $\Delta_3(a,b,-a-b)=4$ there is still only 1 useful GJI for (A,B,C) in a certain order and the new consistent conditions are

$$
\begin{aligned}
&n_{AB}=1,\quad n_{BC}=1,\\
&n_{AC}=1\Rightarrow C_{a,b}C_{a+b,-a-b}(2-\alpha_{a,-a-b}+d_{a,b}\alpha_{a+b,-a-b})+C_{b,-a-b}C_{a,-a}(2-\alpha_{a,b}+d_{b,-a-b}\alpha_{a,-a})\\
&\quad-\mu_{a,b}C_{a,-a-b}C_{b,-b}(2-\alpha_{a,b}+d_{a,-a-b}\alpha_{b,-b})=0.
\end{aligned}
\tag{F15}
$$

For $\Delta_3(a,b,-a-b)\geqslant 5$ there are still no useful GJI's and thus no extra consistent conditions.

c. $\{A,B,C\}=\{\psi_a,\psi_b,\psi_{-b}\}, a\pm b\neq 0 \bmod n$

Now we have $N_{\psi_a,\psi_{\pm b}}=1$, $N_{\psi_b,\psi_{-b}}=2$. For $\Delta_3(a,b,-b)=0$ the consistent conditions are

$$
\begin{aligned}
&h^{\mathrm{sc}}_b h^{\mathrm{sc}}_a=\alpha_{a,\pm b}=0,\quad h^{\mathrm{sc}}_b\partial\psi_a\equiv 0,\\
&C_{a,b}C_{a+b,-b}=\mu_{a,b}C_{a,-b}C_{b,a-b}=C_{b,-b},\\
&(d_{a\pm b,\mp b}-1)\partial\psi_a=(d_{a\pm b,\mp b}-1)\partial\psi_a=d_{\pm b,a}\partial\psi_a=d_{\pm b,a\mp b}\partial\psi_a\equiv 0
\end{aligned}
\tag{F16}
$$

For $\Delta_3(a,b,-b)=2$ the consistent conditions are

$$
\frac{\mu_{a,b}C_{a,-b}C_{b,a-b}}{C_{b,-b}}=\frac{\alpha_{a,b}}{\alpha_{a,b}-d_{a,-b}\alpha_{b,a-b}}=\frac{\alpha_{a,b}(\alpha_{a,b}-1)}{2}+\frac{2h^{\mathrm{sc}}_a h^{\mathrm{sc}}_b}{c},
\tag{F17}
$$

$$
\frac{C_{a,b}C_{a+b,-b}}{C_{b,-b}}=\frac{2-\alpha_{a,b}}{\alpha_{a,-b}-d_{a,b}\alpha_{a+b,-b}}=\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h^{\mathrm{sc}}_a h^{\mathrm{sc}}_b}{c},
\tag{F18}
$$

$$
\frac{C_{a,b}C_{a+b,-b}}{\mu_{a,b}C_{a,-b}C_{b,a-b}}=\frac{\alpha_{a,-b}-2-d_{a,b}\alpha_{a+b,-b}}{\alpha_{a,b}-2-d_{a,-b}\alpha_{b,a-b}},
\tag{F19}
$$

$$
\left\{d_{\pm b,a\mp b}\left[\frac{2h^{\mathrm{sc}}_a h^{\mathrm{sc}}_b}{c}+\frac{\alpha_{a,\pm b}(\alpha_{a,\pm b}-1)}{2}\right]+(1-d_{a\pm b,\mp b})\left[\frac{2h^{\mathrm{sc}}_a h^{\mathrm{sc}}_b}{c}+\frac{(\alpha_{a,\pm b}-2)(\alpha_{a,\pm b}-1)}{2}\right]-\frac{2h^{\mathrm{sc}}_b}{c}\right\}\partial\psi_a\equiv 0,
\tag{F20}
$$

$$
\left\{\left[\frac{2h^{\mathrm{sc}}_a h^{\mathrm{sc}}_b}{c}+\frac{\alpha_{a,\pm b}(\alpha_{a,\pm b}-1)}{2}\right][d_{a,-b}(d_{b,a-b}-1)+1]+\alpha_{a,b}d_{b,a-b}-\frac{2h^{\mathrm{sc}}_b}{c}\right\}\partial\psi_a\equiv 0,
\tag{F21}
$$

$$
\left\{\left[\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h^{\mathrm{sc}}_a h^{\mathrm{sc}}_b}{c}\right](1-d_{a,b}d_{a+b,-b})+(\alpha_{a,b}-2)d_{a+b,-b}-\frac{2h^{\mathrm{sc}}_b}{c}\right\}\partial\psi_a\equiv 0,
\tag{F22}
$$

$$\left\{\left[\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right]d_{a+b,-b}(d_{a,b}-1)+\left[\frac{\alpha_{a,b}(\alpha_{a,b}-1)}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right]d_{b,a-b}+(\alpha_{a,b}-2)(1-d_{a+b,-b})\right\}\partial\psi_a\equiv 0, \tag{F23}$$

$$\left\{\left[\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right]d_{a+b,-b}+\left[\frac{\alpha_{a,b}(\alpha_{a,b}-1)}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right](d_{a,-b}d_{b,a-b}-d_{a,-b}-d_{b,a-b})+\alpha_{a,b}(d_{b,a-b}+1)-1\right\}\partial\psi_a \equiv 0, \tag{F24}$$

$$\left\{\left[\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right](2-d_{a,b})d_{a+b,-b}+\left[\frac{(\alpha_{a,b}-1)\alpha_{a,b}}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right](d_{a,-b}d_{b,a-b}-d_{a,-b}-2d_{b,a-b}) + 1+\alpha_{a,b}d_{b,a-b}+(\alpha_{a,b}-2)d_{a+b,-b}\right\}\partial\psi_a\equiv 0. \tag{F25}$$

We see that central charge c can be determined consistently from the first two conditions. Notice that after a $b\leftrightarrow -b$ exchange the above conditions should also be satisfied.

For $\Delta_3(a,b,-b)=4$ there are four useful GJI's for (A,B,C) in a certain order now and the consistent conditions are

$$C_{a,b}C_{a+b,-b}=\mu_{a,b}C_{a,-b}C_{b,a-b}(\alpha_{b,a-b}d_{a,-b}-\alpha_{a,b}+1)+C_{b,-b}\left[\frac{(\alpha_{a,b}-1)(\alpha_{a,b}-2)}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right], \tag{F26}$$

$$\mu_{a,b}C_{a,-b}C_{b,a-b}=C_{a,b}C_{a+b,-b}(d_{a,b}\alpha_{a+b,-b}-\alpha_{a,-b}+1)+C_{b,-b}\left[\frac{(\alpha_{a,b}-3)(\alpha_{a,b}-2)}{2}+\frac{2h_a^{sc}h_b^{sc}}{c}\right], \tag{F27}$$

$$\left\{C_{a,b}C_{a+b,-b}d_{a+b,-b}(2-\alpha_{a,-b}-d_{a,b}+d_{a,b}\alpha_{a+b,-b})+C_{b,-b}\left[\frac{(\alpha_{a,b}-3)(\alpha_{a,b}-2)}{2}+\frac{2(h_a^{sc}-1)h_b^{sc}}{c}\right] + \mu_{a,b}C_{a,-b}C_{b,-a-b}[d_{b,a-b}(\alpha_{a,b}+d_{a,-b}-2-d_{a,-b}\alpha_{b,a-b})-d_{a,-b}]\right\}\partial\psi_a\equiv 0. \tag{F28}$$

All the above conditions should also hold when we exchange $b\leftrightarrow -b$. For $\Delta_3(a,b,-b)\geqslant 6$ there are no useful GJI's and no extra consistent conditions.

d. $\{A,B,C\}=\{\psi_a,\psi_a,\psi_{-a}\}, a\neq n/2 \bmod n$

Now we have $N_{\psi_a,\psi_a}=1$, $N_{\psi_a,\psi_{-a}}=2$. For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{sc}=0$ the consistent conditions are the same as in Sec. V B,

$$h_a^{sc}=h_{2a}^{sc}=\alpha_{a,a}=0,\quad \partial\psi_a\equiv 0,$$

$$C_{a,a}C_{2a,-a}=C_{a,-a}=C_{-a,a}=\mu_{a,-a}=1. \tag{F29}$$

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{sc}=1$ the consistent conditions are

$$h_{2a}^{sc}=3,\quad h_a^{sc}=1,\quad c=-2,\quad \mu_{a,-a}=-1,$$

$$C_{a,a}C_{2a,-a}=2C_{a,-a},\quad C_{a,-a}=-C_{-a,a},$$

$$\left(d_{2a,-a}-\frac{3}{2}\right)\partial\psi_a=\left(d_{-a,2a}+\frac{1}{2}\right)\partial\psi_a\equiv 0,$$

$$d_{a,a}=1/2. \tag{F30}$$

Notice that $d_{-a,2a}=-1/2$, $d_{2a,-a}=3/2$, and $d_{a,a}=1/2$ are consistent with Eq. (F4) and $h_a^{sc}=1$, $\alpha_{a,a}=-1$.

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{sc}=2$ the consistent conditions are

$$c=\frac{2h_a^{sc}}{3-2h_a^{sc}},\quad \alpha_{a,a}=2-2h_a^{sc},$$

$$C_{a,a}C_{2a,-a}=2h_a^{sc}\neq 0,$$

$$C_{a,-a}=C_{-a,a}=\mu_{a,-a}=1,$$

$$\left(d_{2a,-a}-2+\frac{1}{h_a^{sc}}\right)\partial\psi_a=\left(d_{-a,2a}+1-\frac{1}{h_a^{sc}}\right)\partial\psi_a\equiv 0,$$

$$(2h_a^{sc}-1)d_{a,a}=1/2. \tag{F31}$$

Again notice that Eq. (F4) is consistent with $d_{2a,-a}=2-(h_a^{sc})^{-1}$ and $d_{-a,2a}=(h_a^{sc})^{-1}-1$.

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{sc}=4$ the consistent conditions are

$$\alpha_{a,a} = 4 - 2h_a^{sc}, \quad C_{a,-a} = C_{-a,a} = \mu_{a,-a} = 1,$$

$$C_{a,a}C_{2a,-a} = h_a^{sc}\left(2h_a^{sc} - 3 + \frac{2h_a^{sc}}{c}\right) \neq 0,$$

$$\left(d_{2a,-a} - 2 + \frac{2}{h_a^{sc}}\right)\partial\psi_a = \left(d_{-a,2a} + 1 - \frac{2}{h_a^{sc}}\right)\partial\psi_a \equiv 0,$$

$$(h_a^{sc} - 1)(d_{a,a} - 1/2) = 0. \tag{F32}$$

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{sc}=6$ there is only one useful GJI for (A,B,C) in a certain order now, and the consistent conditions are

$$(2h_a^{sc} - 3)(d_{a,a} - 1/2) = 0,$$

$$(\mu_{a,-a} - 1)\left[\frac{2(h_a^{sc})^2}{c} + (h_a^{sc} - 2)(2h_a^{sc} - 3)\right] = 0. \tag{F33}$$

For $\Delta_3(a,a,-a)=\alpha_{a,a}+2h_a^{sc}\geqslant 7$ we do not have any useful GJI's and there are no consistent conditions.

e. $\{A,B,C\}=\{\psi_{n/2},\psi_{n/2},\psi_{n/2}\}, n=even$

This section is exactly the same as Sec. V B 4 since we still have $N_{AB}=N_{BC}=N_{AC}=2$ if $A=B=C=\psi_{n/2}$. The subleading term in OPE (F1) has no effect on these GJI's.

f. $\{A,B,C\}=\{\psi_a,\psi_b,\sigma_{\gamma+c}\}, a+b\neq 0 \bmod n$

Now we have $N_{\psi_a,\psi_b}=1>0$, so there are new useful GJI's in this case than in Sec. VI D 2. Therefore we have more consistent conditions. For $\Delta_3(a,b,\gamma+c)=0$ the consistent conditions are

$$\mu_{a,b}C_{a,\gamma+c}C_{b,\gamma+a+c} = C_{a,b}C_{a+b,\gamma+c} = C_{b,\gamma+c}C_{a,\gamma+b+c},$$

$$\alpha_{a,\gamma+c} = \alpha_{a+b,\gamma+c}d_{a,b} = \alpha_{c,\gamma+a+b}d_{a,b}. \tag{F34}$$

For $\Delta_3(a,b,\gamma+c)=1$ the consistent conditions are

$$\mu_{a,b}C_{a,\gamma+c}C_{b,\gamma+a+c} = C_{a,b}C_{a+b,\gamma+c}(\alpha_{a,\gamma+c} - d_{a,b}\alpha_{a+b,\gamma+c}),$$

$$C_{b,\gamma+c}C_{a,\gamma+b+c} = C_{a,b}C_{a+b,\gamma+c}(1 - \alpha_{a,\gamma+c} + d_{a,b}\alpha_{a+b,\gamma+c}),$$

$$d_{a,b}(\alpha_{a+b,\gamma+c} - \alpha_{c,\gamma+a+b}) = 0. \tag{F35}$$

For $\Delta_3(a,b,\gamma+c)=2$ the consistent conditions are

$$C_{b,\gamma+c}C_{a,\gamma+b+c} - \mu_{a,b}C_{a,\gamma+c}C_{b,\gamma+a+c} = C_{a,b}C_{a+b,\gamma+c}(\alpha_{a,\gamma+c} - 1 - d_{a,b}\alpha_{a+b,\gamma+c}),$$

$$d_{a,b}(\alpha_{a+b,\gamma+c} - \alpha_{c,\gamma+a+b}) = 0. \tag{F36}$$

For $\Delta_3(a,b,\gamma+c)\geqslant 3$ there are no useful GJI's, and thus no extra consistent conditions.

g. $\{A,B,C\}=\{\psi_a,\psi_{-a},\sigma_{\gamma+c}\}$

This section is exactly the same as Sec. VI D 3 since the subleading term in OPE (F1) has no effect on these GJI's.

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