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# Self-homeomorphisms of 4-manifolds with fundamental group $\mathbb{Z}$

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**Abstract**

In this paper we study the classification of self-homeomorphisms of closed, connected, oriented 4-manifolds with infinite cyclic fundamental group up to pseudoisotopy, or equivalently up to homotopy. We find that for manifolds with even intersection form homeomorphisms are classified up to pseudoisotopy by their action on $\pi_1$, $\pi_2$ and the set of spin structures on the manifold. For manifolds with odd intersection form they are classified by the action on $\pi_1$ and $\pi_2$ and an additional $\mathbb{Z}/2\mathbb{Z}$. As a consequence we complete the classification program for closed, connected, oriented 4-manifolds with infinite cyclic fundamental group, begun by Freedman, Quinn and Wang. 

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## 1. Introduction

The purpose of this paper is to classify, up to pseudoisotopy, the self-homeomorphisms of a closed, oriented, connected, topological 4-manifold with $\pi_1 M = \mathbb{Z}$. This classification was started as a "extended exercise" in [2, Chapter 10] and we follow the outline of that proof. However there are a number of omissions in that argument, several of which effect the conclusion. Both [2] and an earlier version of this paper did not adequately analyze the homeomorphisms over the 1-skeleton. This misses the effect of the homeomorphism on the spin structures. (We are grateful to the referee for pointing out this omission.) Also if $M$ is a closed, oriented, connected, topological 4-manifold with $\pi_1 M = \mathbb{Z}$ and odd intersection form, then the argument in [2] misses a Kervaire–Milnor like obstruction to building a pseudoisotopy over a characteristic class. This extra obstruction was first noted

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in [7] but does not immediately give an obstruction to the existence of a pseudoisotopy. Proving such an obstruction exists is slightly tricky and has not been done until now. As a result, we see that for odd intersection forms there are two pseudoisotopy classes of self-homeomorphisms of $M$ which induce the identity on $\pi_1 M$ and $\pi_2 M$. This fact has been generally regarded as being the case since [7] and [10] first treated the problem.

These results fill in the last detail in the classification results for closed, oriented 4-manifolds with infinite cyclic fundamental group of [2, §10.7]; [10,11] (which includes an extension to nonorientable manifolds.) The final classification result is the following theorem, where the underlined statements are the required changes.

**Theorem 1.**

(1) *Suppose* $(H, \lambda)$ *is a nonsingular hermitian form on a finitely generated free* $\mathbb{Z}[\mathbb{Z}]$-*module* $H$, $k \in \mathbb{Z}/2$, *and if* $\lambda$ *is even then* $k \equiv (\mathrm{signature}\,\lambda)/8 (\mathrm{mod}\, 2)$. *Then there is a closed, oriented* 4-*manifold* $M$ *with* $\pi_1 = \mathbb{Z}$, *intersection form* $\lambda$ *and Kirby–Siebenmann invariant* $k$.

(2) *Suppose* $M$ *and* $M'$ *are two closed, oriented* 4-*manifolds with* $\pi_1 = \mathbb{Z}$, *the same Kirby–Siebenmann invariant,* $h : H_2(M; \mathbb{Z}[\mathbb{Z}]) \to H_2(M'; \mathbb{Z}[\mathbb{Z}])$ *is a* $\mathbb{Z}[\mathbb{Z}]$ *isomorphism that preserves the intersection form, and if the intersection form is even that* $\sigma$ *is a spin structure on* $M$ *and* $\tau$ *is a spin structure on* $M'$. *Then there is a homeomorphism* $f : M \to M'$ *such that* $f_* = h$ *and if* $M$ *has even intersection form* $f^*\tau = \sigma$. *If* $M$ *has even intersection form, then* $f$ *is unique up to pseudoisotopy. If* $M$ *has odd intersection form, then there are exactly two pseudoisotopy classes of such homeomorphisms.*

It should be noted that the classification has also been approached by Kreck using his modified surgery approach [4]. This method gives essentially the same result as above, but this method has not to the authors' knowledge been used to study the classification up to pseudoisotopy of the homeomorphism $f$ in the statement of the theorem.

Phrased purely in terms of self-homeomorphisms the new results in this paper can be described as follows. Suppose $M$ is a closed, oriented, connected topological 4-manifold. Let $(\pi_2 M = H_2(M; \mathbb{Z}[\pi_1 M]), \lambda)$ denote the intersection form of $M$. Let $\mathrm{Aut}(\pi_1 M, \pi_2 M, \lambda)$ denote the group of automorphisms of the pair $(\pi_1 M, \pi_2 M)$ which preserve $\lambda$. That is an element of $\mathrm{Aut}(\pi_1 M, \pi_2 M, \lambda)$ is a pair $(g, \phi)$ where $g : \pi_1 M \to \pi_1 M$ is a group automorphism and $\phi : \pi_2 M \to \pi_2 M$ is a $\mathbb{Z}[\pi_1 M]$ module isomorphism if the $\pi_1 M$ actions are identified via $g$ which preserves $\lambda$. If $\pi_1 M = \mathbb{Z}$ and $\pi_2 M$ is non-trivial we may describe this as the group of $\mathbb{Z}[\mathbb{Z}]$-automorphisms and anti-automorphisms preserving $\lambda$. Let $\mathrm{Spin}(M)$ denote the set of spin structures on $M$ and $S(\mathrm{Spin}(M))$ the group of permutations of $\mathrm{Spin}(M)$. (If $M$ has odd intersection form these are trivial.) Let $\mathrm{TOP}(M)$ denote the group of orientation preserving homeomorphisms of $M$. Let $N \subset \mathrm{TOP}(M)$ be the subgroup of homeomorphisms pseudoisotopic to the identity. Since any homomorphism of $M$ preserves $\lambda$ and any element of $N$ clearly induces the identity map on $\pi_1 M, \pi_2 M$ and $S(\mathrm{Spin}(M))$ there is a natural homomorphism

$$\mathrm{TOP}(M)/N \to \mathrm{Aut}(\pi_1 M, \pi_2 M, \lambda) \times S\big(\mathrm{Spin}(M)\big).$$

In these terms we will show the following theorem.

**Theorem 2.** *Suppose $M$ is a closed, oriented, connected topological 4-manifold with fundamental group $\mathbb{Z}$.*

(a) *If $(H_2(M;\mathbb{Z}[\pi_1 M]),\lambda)$ is even, then the natural homomorphism $\mathrm{TOP}(M)/N \to \mathrm{Aut}(\pi_1 M, \pi_2 M, \lambda) \times S(\mathrm{Spin}(M))$ is an isomorphism.*

(b) *If $(H_2(M;\mathbb{Z}[\pi_1 M]),\lambda)$ is odd, then there is a short exact sequence*

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{TOP}(M)/N \to \mathrm{Aut}(\pi_1 M, \pi_2 M, \lambda) \to 0.$$

These results should be compared to results of Quinn [5] for the simply connected case. He shows the much stronger fact that for simply connected $M$ (regardless of intersection form) the natural homomorphisms give isomorphisms

$$\pi_0\,\mathrm{TOP}(M) \cong \mathrm{TOP}(M)/N \cong \mathrm{Aut}\big(H_2(M;\mathbb{Z}),\lambda\big).$$

Thus in either case for fundamental group $\mathbb{Z}$ we have an extra $\mathbb{Z}/2\mathbb{Z}$. It is not clear whether there is a connection between these two extra factors of $\mathbb{Z}/2\mathbb{Z}$ or in fact whether the exact sequence in (b) is split.

The following proposition is an immediate corollary. It can be proved exactly as in the smooth case as a standard consequence of the surgery exact sequence, compare, for example, [5, Proposition 2.2] or [6, p. 350]. As a consequence the extra self-homeomorphisms are not even homotopic to the identity and $N$ may also be described as the subgroup of homeomorphisms homotopic to the identity.

**Proposition 3.** *Suppose $M$ is a closed, oriented, connected, topological 4-manifold with fundamental group $\mathbb{Z}$, and $f, f' : M \to M$ are two homeomorphisms, then $f$ and $f'$ are pseudoisotopic if and only if $f$ and $f'$ are homotopic.*

This paper is organized as follows. Section 2 contains a brief review of the techniques developed in Freedman and Quinn specialized to the relatively simple case where the fundamental group has no 2-torsion. In Section 3 these results are used to prove Theorem 2 above.

## 2. Review of embedding results

The classification argument given in [2] is based on a clever use of existence and uniqueness theorems for codimension 2 embeddings. An existence result for embeddings in dimension 4 is used for the existence part of Theorem 1 above and an existence result for embeddings in dimension 5 for the uniqueness part. Unfortunately the statements and proofs of these results in [2] contain a number of minor flaws which complicate the classification. Specialized to the case where the fundamental group contains no 2-torsion the corrected versions read as follows.

Consider the following situation. Let $(W, \partial W)$ and $(V, \partial_0 V, \partial_1 V)$ be topological 4-manifolds with boundary. We wish to consider the case where $V$ looks homotopically like it is built from $\partial_0 V$ by adding 0-handles and 2-handles. Specifically suppose $\pi_1(V, \partial_0 V) = \pi_1(V, \partial_1 V) = 1$, each component of $V$ has nonempty intersection with $\partial_1 V$ and components disjoint from $\partial_0 V$ are 1-connected. Suppose we are given a map $h : V \to W$ which restricts to an embedding $\partial_0 V \to \partial W$ and preserves all intersection and self-intersection numbers, including the relative intersection and self-intersection numbers for classes with boundary in $\partial_0 V$. (For an abstract definition of these relative intersection numbers see [2, §10.5].) We wish to alter $h$ to produce a $\pi_1$-negligible embedding of $V$ in $W$ and to classify such embeddings up to $\pi_1$-negligible concordance. Therefore we also need algebraic dual 2-spheres in $W$ to the 2-handles of $h(V)$. See [2, §10.5] for a more algebraic statement of this condition. Call a map $h$ satisfying all of the above conditions a $\pi_1$-negligible embedding problem $h : (V, \partial_0 V, \partial_1 V) \to (W, \partial W)$.

For a $\pi_1$-negligible embedding problem, with some extra fundamental group assumptions, [2, Chapter 10] and [7] give a short list of obstructions to homotoping $h \operatorname{rel} \partial_0 V$ to a $\pi_1$-negligible embedding and a short list of obstructions to finding a $\pi_1$-negligible concordance between two such embeddings. The uniqueness part of this result can be phrased as follows. Call a $\pi_1$-negligible embedding problem $s$-characteristic if $\omega_2 : \pi_2 W \to \mathbb{Z}/2\mathbb{Z}$ does not vanish but does vanish on the subspace of $\pi_2 W = H_2(W; \mathbb{Z}[\pi_1 W])$ perpendicular (in the sense of intersection pairings) to $h_* H_2(V, \partial_0 V; \mathbb{Z}[\pi_1 W])$. Paraphrased, $h$ is $s$-characteristic if the universal cover $\widetilde{W}$ of $W$ is not spin, but for some element $x \in H_2(V, \partial_0 V; \mathbb{Z}[\pi_1 W]), h_*(x)$ is characteristic in $\widetilde{W}$.

**Theorem 4** (Freedman, Quinn and Stong). *Suppose $h : (V, \partial_0 V, \partial_1 V) \to (W, \partial W)$ is a $\pi_1$-negligible embedding problem and $\pi_1 W$ is "good" and contains no 2-torsion. Suppose $f_1$ and $f_2$ are two $\pi_1$-negligible embeddings homotopic* $\operatorname{rel} \partial_0 V$ *to $h$. Fix a homotopy $H \operatorname{rel} \partial_0 V$ from $f_1$ to $f_2$.*

(a) *If $h$ is not $s$-characteristic, then $H$ is homologous, with $\mathbb{Z}[\pi_1 W]$-coefficients, to a $\pi_1$-negligible concordance between $f_1$ and $f_2$.*
(b) *If $h$ is $s$-characteristic, then there is an obstruction* $\operatorname{km}(H) \in H_1(W; \mathbb{Z}/2\mathbb{Z})$ *which vanishes if and only if $H$ is homologous, with $\mathbb{Z}(\pi_1 W)$-coefficients, to a $\pi_1$-negligible concordance between $f_1$ and $f_2$.*
(c) *If $h$ is $s$-characteristic, $f_1$ is fixed and $\alpha \in H_1(W; \mathbb{Z}/2\mathbb{Z})$, then there is a $\pi_1$-negligible embedding $f_2$ and a homotopy $H \operatorname{rel} \partial_0 V$ from $f_1$ to $f_2$ such that* $\operatorname{km}(H) = \alpha$.

The proof of this theorem is contained in [2] and [7] and will not be reproduced here. However a few of the details deserve comment. The term "good" group refers to the groups for which topological surgery in dimension 4 works. Freedman [1] showed that $\mathbb{Z}$ is good which is all we need for this paper. He in fact also showed that all the elementary amenable groups are good. By recent results of Freedman and Teichner all groups of subexponential growth are good [3]. It is still open whether all groups are good, but it is generally believed that the free group on $n$ generators, $n \geqslant 2$, is not good.

An exact description of the invariant km will be needed for the proof below. In the easy case of Theorem 4 [7] contains a usable combinatorial description. Choose a class $x \in H_2(V, \partial_0 V; \mathbb{Z}[\pi_1 W])$ for which $h_*(x)$ is characteristic in $\widetilde{W}$. After possibly stabilizing $W$ and $V$ by adding copies of $S^2 \times S^2$ we may choose an embedded 2-disk $(D^2, S^1) \to (V, \partial_0 V)$ representing $x$. Then we may view $H|_{(D^2,S^1)\times I}$ as an immersion $G:(D^3, S^2) \to W$ which is an embedding near the boundary. Put this immersion into general position. Then the singular set $\Sigma_G$ of $G$ forms a link in $D^3$. The components of this link fall into three classes. The first are pairs of circles which map under $G$ to a single circle in $W$. The other two are circles which under $G$ double cover either a circle in $W$ or an arc in $W$ with cusps at the endpoints. Consider a pair of circles $C$ and $C'$ which cover the same circle in $W$. Then one can assign an element of $H_1(W; \mathbb{Z}/2\mathbb{Z})$ as follows. Choose points $x \in C$ and $x' \in C'$ which map to the same point of $W$. Let $\gamma$ be a path in $D^3$ from $x$ to $x'$. Then $G(\gamma)$ is a closed loop in $W$ which represents the desired element of $H_1(W; \mathbb{Z}/2\mathbb{Z})$. Denote this element by $a_C$. Let $\mathrm{lk}(C)$ be the number of components of $\Sigma_G - C$ linked by $C$ counted mod 2. (By [7] this is the same as $\mathrm{lk}(C')$, hence the apparent asymmetry in our definition is only apparent.) Then

$$\mathrm{km}(H) = \sum_{\{C,C'\}} \mathrm{lk}(C) a_C,$$

where the sum runs over all pairs of circles in $\Sigma_G$ which cover a single circle in $W$. Note that from this definition it is clear that $\mathrm{km}(H)$ is unchanged if we stabilize $W$ by adding a connected sum with a closed, 1-connected, spin 4-manifold.

## 3. Applications to self-homeomorphisms

We now turn to the proof of Theorem 2 above. Let $M$ be a closed, oriented, connected topological 4-manifold with fundamental group $\mathbb{Z}$. First we will show surjectivity of the natural map $\mathrm{TOP}(M)/N \to \mathrm{Aut}(\pi_1 M, \pi_2 M, \lambda) \times S(\mathrm{Spin}(M))$. Suppose we are given a group automorphism $g:\pi_1 M \to \pi_1 M$ and a $\mathbb{Z}(\pi_1 M)$ module isomorphism $\phi:\pi_2 M \to \pi_2 M$. If $M$ is even, fix a spin structure $\sigma$ on $M$ and let $\tau$ be another spin structure on $M$ (possible the same one). Choose an embedding circle $\gamma$ in $M$ representing the generator of $\pi_1 M$. Then a closed regular neighborhood $N$ of $\gamma$ is homeomorphism to $S^1 \times D^3$. If $M$ is even, choose this trivialization to agree with the one given by $\sigma$ otherwise fix any trivialization. We wish to start building our homeomorphism $f$ with a homeomorphism $f|_N: N \to N$. If $g$ is the identity make $f|_N$ the identity on the core $S^1$, otherwise make it the reverse. If $M$ is odd extend it to $N$ arbitrarily. If $M$ is even, then the spin structure $\tau$ gives another trivialization of $N$. Use this trivialization to build $f|_N$. Let $M_S$ be $M - \mathrm{int}(N)$. Choose another embedded circle $\gamma'$ in $\mathrm{int}(M_S)$ representing the generator of $\pi_1 M_S = \pi_1 M$ and let $E$ be a closed regular neighborhood of $\gamma'$ let $V$ be $M_S - \mathrm{int}(E)$ and let $\partial_0 V = \partial M_S = \partial N$. Then $\phi:\pi_2 M \to \pi_2 M$ determines, up to homotopy, a map $h:(V, \partial_0 V) \to (M_S, \partial_0 V)$ extending $f|_{\partial N}$. One easily checks that homotopically (in fact geometrically after stabilization by connected sum with $E_8$ and $S^2 \times S^2$) $V$ is built from $\partial_0 V$ by adding 2-handles. Also for any isomorphism

$$\phi: H_2\big(V, \partial_0 V; \mathbb{Z}[\mathbb{Z}]\big) \to H_2\big(M_S, \partial M_S; \mathbb{Z}[\mathbb{Z}]\big)$$

Poincaré duality guarantees the existence of dual 2-spheres. Therefore by Theorem 10.5 of [2] $h$ is homotopic to a $\pi_1$-negligible embedding. Use this to extend $f$ over $V$ to a map $f|_{M-\mathrm{int}(E)} : M - \mathrm{int}(E) \to M$. Fix a trivialization of $E$. Then $M - f(M - E)$ is a homotopy $S^1$ with a fixed identification of its boundary with $S^1 \times S^2$. By [2, Proposition 11.6A] there is a homeomorphism $S^1 \times D^3 \to M - f(M - E)$ extending the given one on the boundary. This extends $f$ to a homeomorphism $f : M \to M$. Note that $f$ induces the given maps $g$ and $\phi$ on $\pi_1$ and $\pi_2$ and in the even case takes $\sigma$ to $\tau$.

Now we wish to apply Theorem 4 above to study the kernel of the homomorphism. Let $M, \gamma, N, \gamma', E$ and $V$ be as above. Further suppose that $f : M \to M$ is a homeomorphism which induces the identity on $\pi_1, \pi_2$ and in the even case on the spin structures on $M$. We want to start building a pseudoisotopy $F$ from the identity to $f$. However applying Theorem 4 to classify self-homeomorphisms up to pseudoisotopy is slightly subtle for two reasons. That theorem essentially describes what happens to the 2-skeleton. However it is not clear that obstructions to extending a pseudoisotopy defined over the 1-skeleton to one defined over the 2-skeleton actually give obstructions to finding a pseudoisotopy. Further the obstruction $\mathrm{km}(H)$ to uniqueness of embeddings depend (weakly) on the choice of a homotopy $H$ between the two embeddings. The purpose of this section is to resolve these issues.

Clearly to show $f$ is pseudoisotopic to the identity, it suffices to build a pseudoisotopy $F : M \times I \to M \times I$ from id to any homeomorphism homotopic to $f$. Thus we are free to change $f$ by an isotopy whenever we desire. We will continue to denote this modified map by $f$. By a first isotopy we may assume $f$ is the identity on $\gamma$. For such an $f$ we may take $F$ to be $\mathrm{id} \times I$ on $\gamma \times I$. We have a trivialization of $N \times \{0\}$ as $S^1 \times D^3$ and applying $f$ gives a trivialization of $N \times \{1\}$. In the even case, since $f$ preserves spin structures this is the isotopic to the trivialization on $N \times \{0\}$ and after isotoping $f$ we can extend $F$ to be the identity on $N \times I$. If $M$ is odd, then $M$ is not spin and we may isotop $f$ so that $f$ induces the same trivialization of $N$ as the identity. Hence in this case we may also extend $F$ over $N \times I$ to be the identity. Next we wish to extend $F$ over $(M_S - \mathrm{int}(E)) \times I$. The problem of extending $F$ over $(M_S - \mathrm{int}(E)) \times I$ is exactly the problem solved by Theorem 4. If $\lambda$ is even, there is no obstruction to extending $F$ by Theorem 4(a). If $\lambda$ is odd and if one fixes a homotopy $H$ between $\mathrm{id}|_{M_S-\mathrm{int}(E)}$ and $f|_{M_S-\mathrm{int}(E)}$, then by Theorem 4(b) there is an obstruction $\mathrm{km}(H) \in H_1(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ to finding a $\pi_1$-negligible concordance between $\mathrm{id}|_{M_S-\mathrm{int}(E)}$ and $f|_{M_S-\mathrm{int}(E)}$ homologous (with $\mathbb{Z}[\pi_1 M]$ coefficients) to $H$. If this obstruction vanishes (for some choice of $H$), then we can extend $F$ over $(M_S - \mathrm{int}(E)) \times I$. In either of the cases where we can extend $F$, we have defined $F : (M - \mathrm{int}(E)) \times I \to M \times I$. The complement of $\mathrm{im}(F|_{(M-E)\times I})$ is homeomorphic to $S^1 \times D^4$ and has a fixed identification of its boundary with $S^1 \times S^3$. By [2, Theorem 11.5] $S^1 \times D^4$ is unique up to homeomorphism rel boundary. Therefore $F$ extends to a pseudoisotopy from id to $f$.

This completes the proof in the case where $M$ is even. In the odd case, the obstruction $\mathrm{km}(H)$ is additive under disjoint unions in the following sense. If $H$ is a homotopy from $f_1$ to $f_2$ and $H'$ is a homotopy from $f_2$ to $f_3$, then we may regard $H \cup H'$ as a homotopy from $f_1$ to $f_3$. Since km is calculated by adding up contributions from the self-intersections of the homotopy one clearly has $\mathrm{km}(H \cup H') = \mathrm{km}(H)\,\mathrm{km}(H')$. Therefore in the odd case we have two possibilities for the kernel of $\mathrm{TOP}(M)/N \to \mathrm{Aut}(\pi_1 M, \pi_2 M, \lambda)$, either $\mathbb{Z}/2\mathbb{Z}$ or 1. The former case occurs if and only if km is actually independent of the choice of homotopy and some homeomorphism realizes the nonzero value of km.

In the case where $\lambda$ is odd, it is possible to find a self-homeomorphism $f$ of $M$ which is the identity on $N$ and induces the identity on $\pi_2 M$ and for which some homotopy realizes the nontrivial value of this obstruction km. For this alter the construction of self-homeomorphism of $M$ given above to exploit Theorem 4(c). When it comes time to extend the homeomorphism $h$ over $V$ instead of using the identity, we argue that by Theorem 4(c) there is an embedding $f : V \to M_S$ and a homotopy $H$ rel $\partial_0 V$ from $\mathrm{id}|_V$ to $f$ with $\mathrm{km}(H)$ nonzero. To extend $f$ over $E$ we proceed as above. $M_S - \mathrm{int}(f(V))$ is a homotopy circle with boundary identified with $S^1 \times S^2$. Hence $f$ extends to a homeomorphism $f : M \to M$.

It only remains to show that $\mathrm{km}(H)$ is independent of the choice of the homotopy $H$. First assume that $M = S^1 \times S^3 \# N$ where $N$ is a closed, oriented, 1-connected, topological 4-manifold with odd intersection form. Suppose further $N$ has a characteristic class represented by an locally flat, topologically embedded 2-sphere $S \subset N$. Suppose $H : S^2 \times I \to M \times I$ is the homotopy from $S$ to $f(S)$. Up to homotopy any other homotopy $H'$ from $S$ to $f(S)$ differs from $H$ by forming an ambient connected sum with an element of $\pi_3(M \times I)$. Since the invariant $\mathrm{km}(H')$ depend only on the homology class of $H'$, we need only calculate $\mathrm{km}(H')$ for one element of $\pi_3(M \times I)$ in each class in $H_3(M \times I; \mathbb{Z}[\pi_1 M])$. But

$$H_3\big(M \times I; \mathbb{Z}[\pi_1 M]\big) \cong H_3\big(M; \mathbb{Z}[\pi_1 M]\big) \cong \mathbb{Z}$$

and is generated by the core $S^3$ in $S^1 \times S^3$. Thus it suffices to show that $\mathrm{km}(H)$ is unchanged if we form an ambient connected sum of $H$ with a copy of this 3-sphere. We may assume $H$ is a product $S \times [0, \varepsilon)$ inside $M \times [0, \varepsilon)$ and we may choose our 3-sphere to be the obvious one in $M \times \{\varepsilon/2\}$ which is embedded disjoint from $\mathrm{im}(H)$. Perform the ambient connected sum in a neighborhood of an arc from $\mathrm{im}(H)$ to this 3-sphere with interior disjoint from both. Then the resulting homotopy $H'$ has exactly the same self-intersections as $H$. Since $\mathrm{km}(H')$ is computed entirely from linking numbers of self-intersection components we must have $\mathrm{km}(H') = \mathrm{km}(H)$.

For the general case it does not seem to be possible to compute the effect of changing the homotopy $H$ directly. This difficulty is easy to get around by stabilization. The invariant km is unchanged if one takes a connected sum with $S^2 \times S^2$. Therefore it suffices to show that one can get to the case above by such stabilizations. Suppose $M$ is any closed, oriented, connected, topological 4-manifold with $\pi_1 = \mathbb{Z}$ and odd intersection form. Then by the stable classification of such manifolds [4,10] there is some positive integer $k$, such that $M \# k(S^2 \times S^2)$ is homeomorphic to $S^1 \times S^3 \# N$ for some $N$. (By a recent example of Teichner's [9] there are examples $M$ which are not of the form $S^1 \times S^3 \# N$.) If necessary adding another copy of $S^2 \times S^2$, we may assume $N$ has an indivisible, characteristic class $\alpha$ with

$$\alpha \cdot \alpha \equiv \mathrm{sign}(N) + 8KS(N) (\mathrm{mod}\, 16).$$

By [2, §10.8] $\alpha$ is represented by a locally flat embedded 2-sphere.

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