Lectures on Differential Geometry
Math 240A
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Preface

This is a set of lecture notes for the course Math 240A given during the Fall of 2014. The notes will evolve as the course progresses. Starred sections are not fully covered in the lectures; they discuss digressions which are less central to the core subject matter of the course.

The bibliography at the end of the notes provides some suggestions for further reading related to the topics treated in the course.
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Chapter 1

Smooth manifolds

Manifolds are higher dimensional generalizations of curves and surfaces in Euclidean space.

It is important to develop “calculus on manifolds” for many reasons. For example, to provide a mathematical formulation for heat flow on a surface in Euclidean space, we need a theory of partial differential equations on a curved surface. More generally, we need a global calculus of several variables to properly formulate problems arising in subjects such as electricity and magnetism, fluid mechanics, classical mechanics or general relativity. Such a calculus of several variables extends and clarifies Stokes’s Theorem and the Divergence Theorem from ordinary calculus of several variables. Finally, manifolds provide an important family of examples to which we can apply the techniques of topology.

1.1 Topological manifolds

The first step is the theory of topological manifolds, and for this, we need to assume some background in real analysis. Although all we really need is the topology of metric spaces, the more general notion of topological space is sometimes useful. Rudin [16] is a good reference for real analysis with far more than we need, while Munkres [15] provides a nice introduction to point-set topology with all the key definitions and examples.

Definition. A topological space is a pair \((X, \mathcal{T})\) where \(X\) is a set and \(\mathcal{T}\) is a collection of subsets of \(X\) such that

1. the empty set \(\emptyset\) and the entire space \(X\) are both elements of \(\mathcal{T}\),
2. the intersection of any two elements in \(\mathcal{T}\) is in \(\mathcal{T}\), and
3. the union of an arbitrary collection of elements in \(\mathcal{T}\) lies in \(\mathcal{T}\).

We say that \(\mathcal{T}\) is a topology on \(X\). Elements of \(\mathcal{T}\) are called open sets. A subset \(A \subseteq X\) is said to be closed if its complement \(X - A\) is open.
**Key example.** The example needed most for multivariable calculus is $X = \mathbb{R}^n$ with the topology defined by the standard metric

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \text{with} \quad d(x, y) = |x - y|.$$  

Using this metric, we define the $\varepsilon$-ball $N(x; \varepsilon)$ about a point $x \in X$ to be

$$N(x; \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$  

We say that a subset $U$ of $X$ lies in the topology $T$ defined by the metric $d$ if

$$x \in U \implies N(x; \varepsilon) \subseteq U, \quad \text{for some } \varepsilon > 0.$$  

One can then check that this collection $T$ of subsets of $X$ satisfies all the above axioms for topological space. For each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$, $N(x; \varepsilon)$ is open, and we call $N(x; \varepsilon)$ an *open ball* about $x$ of radius $\varepsilon$. We say that the $T$ we have just constructed is the *standard topology* on $\mathbb{R}^n$.

**Another example.** Suppose that $X$ is a subset of $\mathbb{R}^n$. In this case, we can let $T$ be the collection of subsets of $X$ such that

$$U \subseteq T \iff U = V \cap X, \quad \text{where } V \text{ is open in } \mathbb{R}^n.$$  

Then $T$ is the *subspace topology* that $X$ inherits from $\mathbb{R}^n$.

**Definition.** Suppose that $(X_1, T_1)$ and $(X_2, T_2)$ are topological spaces. A function $f : X_1 \to X_2$ is *continuous* if

$$U_2 \in T_2 \implies f^{-1}(U_2) \in T_1.$$  

In other words, a function $f : X_1 \to X_2$ is continuous if the inverse image of any open set in $X_2$ is open in $X_1$. A function $f : X_1 \to X_2$ is said to be a *homeomorphism* if it is a continuous bijection which has a continuous inverse $f^{-1} : X_2 \to X_1$.

A topologist would regard two topological spaces $(X_1, T_1)$ and $(X_2, T_2)$ as essentially the same if there is a homeomorphism from $(X_1, T_1)$ to $(X_2, T_2)$. One of the fundamental problems of topology would be to classify all topological spaces up to homeomorphism, but it is impossibly difficult. We are more interested in understanding those topological spaces on which one could do calculus. There are two additional axioms of a somewhat technical nature that we will impose to exclude pathological topological spaces.

**Definition.** A topological space $(X, T)$ is *Hausdorff* if whenever $p$ and $q$ are distinct elements of $X$, there exist elements $U, V \in T$ such that

$$p \in U, \quad q \in V, \quad U \cap V = \emptyset.$$  

Note that $\mathbb{R}^n$ with its standard topology is Hausdorff. Indeed, if $x$ and $y$ are distinct elements of $\mathbb{R}^n$, we can set $\varepsilon = d(x, y)$. If

$$U = N(x; \varepsilon/3) \quad \text{and} \quad V = N(y; \varepsilon/3),$$

then $U$ and $V$ are disjoint open sets containing $x$ and $y$, respectively.
Then
\[ x \in N(x; \varepsilon/3), \quad y \in N(y; \varepsilon/3), \quad N(x; \varepsilon/3) \cap N(y; \varepsilon/3) = \emptyset, \]
and hence \( \mathbb{R}^n \) with its standard topology is indeed Hausdorff.

**Definition.** Suppose that \((X, \mathcal{T})\) is a topological space. A subset \( \mathcal{B} \subseteq \mathcal{T} \) is a **countable base** for \( \mathcal{T} \) if \( \mathcal{B} \) is countable and every element of \( \mathcal{T} \) is a union of elements of \( \mathcal{B} \). If the topological space \((X, \mathcal{T})\) has a countable base, it is said to be **second countable**.

In the case of \( \mathbb{R}^n \) with its standard topology, we can let \( \mathcal{B} \) be the set of open balls
\[ N((r_1, r_2, \ldots, r_n), \varepsilon) \text{ such that } r_1, r_2, \ldots, r_n, \varepsilon \in \mathbb{Q}, \]
then \( \mathcal{B} \) is countable and one can check that any open set is a union of elements of \( \mathcal{B} \). Thus \( \mathbb{R}^n \) with its standard topology is second countable.

It is easy to verify that any subspace of \( \mathbb{R}^n \) with the subspace topology is also Hausdorff and second countable.

**Definition.** An **\( n \)-dimensional topological manifold** is a Hausdorff second countable topological space \((M, \mathcal{T})\) such that every point \( p \in M \) lies in an open set which is homeomorphic to an open subset of \( \mathbb{R}^n \) with the topology induced from the standard topology on \( \mathbb{R}^n \).

Alternatively, we can say that \( M \) is a **topological manifold of dimension \( n \)**.

### 1.2 Smooth manifolds

Here is some additional useful terminology for topological manifolds: A **chart** or **coordinate system** on an \( n \)-dimensional topological manifold \( M \) is a pair \((U, \phi)\), where \( U \) is an open subset of \( M \) and \( \phi \) is a homeomorphism from \( U \) onto an open subset \( \phi(U) \) of \( \mathbb{R}^n \).

An **atlas** \( \mathcal{A} \) on \( M \) is a collection \( \mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\} \) of charts on \( M \) such that
\[ \bigcup\{U_\alpha : \alpha \in A\} = M. \]

We say that an atlas \( \mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\} \) on \( M \) is **smooth** if
\[ \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \]
is smooth (that is, \( C^\infty \)) for every choice of \( \alpha, \beta \in A \). Two smooth atlases \( \mathcal{A} \) and \( \mathcal{B} \) on \( M \) are said to be **equivalent** if \( \mathcal{A} \cup \mathcal{B} \) is a smooth atlas.

**Definition.** An **\( n \)-dimensional smooth manifold** or **a smooth manifold of dimension \( n \)** is an \( n \)-dimensional topological manifold together with an equivalence class of smooth atlases.
Suppose now that $M$ and $N$ are smooth manifolds of possibly different dimensions) with smooth atlases $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ and $\mathcal{B} = \{(V_\beta, \phi_\beta) : \beta \in B\}$ respectively. Then a function $F : M \to N$ is said to be smooth if

$$
\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \to \psi_\beta(V_\beta)
$$

is smooth for each $\alpha \in A$ and each $\beta \in B$.

Note that the identity map $\text{id} : M \to M$ from a manifold to itself is a smooth map, and that if $F : M \to N$ and $G : N \to P$ are smooth maps between smooth manifolds, then so is the composition $G \circ F : M \to P$. We say smooth manifolds and maps form a “category” with the “objects” being the smooth manifolds and the “morphisms” being smooth maps. A smooth function $F : M \to N$ is a diffeomorphism if it is a bijection and its inverse $F^{-1} : N \to M$ is smooth. Two diffeomorphic manifolds are often regarded as “the same” from the point of view of differential topology.

**Example 1.** The simplest example of an $n$-dimensional smooth manifold is $\mathbb{R}^n$. In this case, we can take the atlas to consist of a single element, $\mathcal{A} = \{(\mathbb{R}^n, \text{id})\}$, where $\text{id}$ denotes the identity map on $\mathbb{R}^n$. Any open subset of $\mathbb{R}^n$ is also a smooth manifold.

Two-dimensional smooth manifolds are also called smooth surfaces, and the first rigorous definition of smooth manifold seems to have occurred within Weyl’s treatment of the theory of Riemann surfaces, a key example of which is the Riemann sphere:

**Example 2.** The Riemann sphere, can be thought of intuitively as the space obtained from the complex plane $\mathbb{C}$ by adding a point at infinity, a point which we denote by $\infty$. To be more rigorous, we regard it as the unit sphere

$$
S^2 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\},
$$

and give it the topology that it inherits as a subspace of $\mathbb{R}^3$. With the subspace topology, it is Hausdorff and second countable and we will shortly see that it is also a topological manifold. In fact, we will construct charts on the Riemann sphere via stereographic projection.

If $N = (0, 0, 1)$ is the north pole on $S^2$, stereographic projection is a one-to-one onto function

$$
\Phi : \mathbb{C} \to S^2 \setminus \{N\},
$$

where $\mathbb{C}$ is thought of as the $(x^1, x^2)$-plane in $\mathbb{R}^3$. It is constructed by considering the line $L$ containing $N = (0, 0, 1)$ on $S^2$ and the point $(x, y, 0)$ in the plane $x^3 = 0$. This line $L$ can be parametrized by

$$
\gamma : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad \gamma(t) = (x^1(t), x^2(t), x^3(t)) = t(x, y, 0) + (1 - t)(0, 0, 1).
$$

Alternatively, we can write this as

$$
x^1(t) = tx, \quad x^2(t) = ty, \quad x^3(t) = 1 - t. \quad (1.1)
$$
There is a unique nonzero value for $t$ such that $\gamma(t)$ lies on $S^2 - \{N\}$, and it occurs when

$$1 = (x^1)^2 + (x^2)^2 + (x^3)^2 = t^2 x^2 + t^2 y^2 + (1 - t)^2.$$ 

We can expand and solve for $t$:

$$1 = t^2(x^2 + y^2) + 1 - 2t + t^2 = t^2(x^2 + y^2 + 1) - 2t + 1,$$

$$2t = t^2(x^2 + y^2 + 1), \quad 2 = t(x^2 + y^2 + 1), \quad t = \frac{2}{x^2 + y^2 + 1}.$$ 

Substitution into (1.1) then yields the point on $S^2$ which corresponds to the point $(x, y, 0)$, corresponding to $x + iy \in \mathbb{C}$:

$$x^1 = \frac{2x}{x^2 + y^2 + 1}, \quad x^2 = \frac{2y}{x^2 + y^2 + 1}, \quad x^3 = 1 - \frac{2}{x^2 + y^2 + 1}. $$

We can then simplify this to

$$x^1 = \frac{2x}{x^2 + y^2 + 1}, \quad x^2 = \frac{2y}{x^2 + y^2 + 1}, \quad x^3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}, \quad (1.2)$$

which gives an explicit formula for stereographic projection

$$\Phi : \mathbb{C} \rightarrow S^2 - \{N\}, \quad \Phi(z) = \left(\frac{2\text{Re}(z)}{|z|^2 + 1}, \frac{2\text{Im}(z)}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right),$$

where $\text{Re}(z)$ and $\text{Im}(z)$ are the real and imaginary parts of $z = x + iy$.

To see that $\Phi$ is one-to-one and onto, we can construct an explicit inverse. Indeed, eliminating $t$ from equations (1.1) gives first $t = 1 - x_3$ and then

$$x = \frac{x^1}{t} = \frac{x^1}{1 - x^3}, \quad y = \frac{x^2}{t} = \frac{x^2}{1 - x^3}. $$

Thus we can set $U = S^2 - \{N\}$ and define a function

$$\phi : U \rightarrow \mathbb{C} \quad \text{by} \quad \phi(x^1, x^2, x^3) = \frac{x^1 + ix^2}{1 - x^3}, \quad (1.3)$$

which is exactly the inverse of $\Phi$. The inverse map $\phi$ is well-behaved except when $x_3 = 1$, that is, it is well-behaved except at the north pole $N$ on $S^2$.

These explicit formulae may seem confusing at first. The important point to note is that as the modulus of $z$ gets larger and larger, $\Phi(z)$ approaches the north pole on $S^2$. Thus if we think of using $\Phi$ to identify points of $\mathbb{C}$ with points on $S^2 - \{N\}$, then the north pole $N$ should be identified with a point at infinity. Indeed we might think of extending $\Phi$ to a map

$$\hat{\Phi} : \mathbb{C} \cup \{\infty\} \rightarrow S^2$$

which is well-behaved.
which takes $\infty$ to the north pole $N$. This idea of adding a point at infinity to the complex plane, thereby obtaining what is sometimes called the extended complex plane or the one-point compactification of $\mathbb{C}$, has turned out to be extremely useful in understanding functions of a complex variable.

On the other hand, we can think of $(U, \phi)$ as defining a chart or a coordinate $z$ on $S^2$, which is well-behaved everywhere except at the north pole $N$. But sometimes we want a coordinate $w$ that is well-behaved near $\infty$, and we might try to take

$$w = \frac{1}{z}$$

as such a coordinate. Can we think of this also as a coordinate on part of $S^2$?

We find that

$$z = \frac{x^1 + ix^2}{1 - x^3} \Rightarrow \quad w = \frac{1}{z} = \frac{1 - x^3}{x^1 + ix^2} = \frac{(1 - x^3)(x^1 - ix^2)}{(x^1)^2 + (x^2)^2}$$

$$= \frac{(1 - x^3)(x^1 - ix^2)}{1 - (x^3)^2} = \frac{x^1 - ix^2}{1 + x^3}$$

Thus if we let $S$ denote the south pole $(0, 0, -1)$, set $V = S^2 - \{S\}$ and define

$$\psi : V \rightarrow \mathbb{C} \quad \text{by} \quad w = \psi(x^1, x^2, x^3) = \frac{x^1 - ix^2}{1 + x^3}. \quad (1.4)$$

It is relatively easy to show that conjugation followed by the inverse of $\psi$ is just stereographic projection with the north pole replaced by the south pole.

We now have two charts $(U, \phi)$ and $(V, \psi)$ defined by (1.3) and (1.4),

$$\phi : S^2 - \{N\} \rightarrow \mathbb{C} \quad \text{and} \quad \psi : S^2 - \{S\} \rightarrow \mathbb{C},$$

which are related by

$$\psi \circ \phi^{-1}(z) = w = \frac{1}{z} \quad \text{and} \quad \phi \circ \psi^{-1}(w) = z = \frac{1}{w}, \quad (1.5)$$

or

$$\psi \circ \phi^{-1}(u^1, u^2) = \frac{(u^1, -u^2)}{(u^1)^2 + (u^2)^2} \quad \text{and} \quad \phi \circ \psi^{-1}(v^1, v^2) = \frac{(v^1, -v^2)}{(v^1)^2 + (v^2)^2}.$$ 

These formulae show that

$$\mathcal{A} = \{(U, \phi), (V, \psi)\}$$

form a smooth atlas on $S^2$, which make it into a smooth two-dimensional manifold.

The Riemann sphere is more than just a two-dimensional smooth manifold. Indeed, the transformation formulae (1.5) make $S^2$ into what is called a Riemann surface, in accordance with the definition we next give.
We say that an atlas \( \mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\} \) on a two-dimensional topological manifold \( M \) is holomorphic if the charts \( (U_\alpha, \phi_\alpha) \) in the atlas take values in the complex plane \( \mathbb{C} \) (which of course is homeomorphic to \( \mathbb{R}^2 \)) and

\[
\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)
\]

is holomorphic for every choice of \( \alpha, \beta \in A \). Two holomorphic atlases \( \mathcal{A} \) and \( \mathcal{B} \) on \( M \) are said to be equivalent if \( \mathcal{A} \cup \mathcal{B} \) is a holomorphic atlas. We can then modify the definition of smooth manifold:

**Definition.** A Riemann surface is a two-dimensional topological manifold together with an equivalence class of holomorphic atlases.

The transformation formulae (1.5) show that the atlas \( \mathcal{A} = \{(U, \phi) \}, \mathcal{B} = \{(V, \psi) \} \) we just constructed on \( S^2 \) is a holomorphic atlas, which makes \( S^2 \) into a Riemann surface.

Suppose that \( \Sigma_1 \) and \( \Sigma_2 \) are Riemann surfaces with holomorphic atlases \( \mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\} \) and \( \mathcal{B} = \{(V_\beta, \phi_\beta) : \beta \in B\} \) respectively. Then a function \( F : \Sigma_1 \rightarrow \Sigma_2 \) is said to be holomorphic if

\[
\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap F^{-1}(V_\beta)) \longrightarrow \psi_\beta(V_\beta)
\]

is holomorphic for each \( \alpha \in A \) and each \( \beta \in B \). A holomorphic function \( F : \Sigma_1 \rightarrow \Sigma_2 \) is a holomorphic diffeomorphism if it is a bijection and its inverse \( F^{-1} : \Sigma_2 \rightarrow \Sigma_1 \) is also holomorphic. One of the goals of Riemann surface theory is to classify Riemann surfaces up to holomorphic diffeomorphism.

**Digression.** The notion of Riemann surfaces makes it much easier to develop the properties of rational functions in complex analysis. Recall that a rational function is simply a complex-valued function of the form

\[
f(z) = \frac{P(z)}{Q(z)}, \tag{1.6}
\]

where

\[
P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0
\]

and

\[
Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0
\]

are polynomials with complex coefficients

\[
a_m, a_{m-1}, \ldots, a_1, a_0, \quad b_n, b_{n-1}, \ldots, b_1, b_0,
\]

and we make the assumptions that \( a_m \neq 0 \) and \( b_n \neq 0 \), and that \( P \) and \( Q \) do not have any common factors. In ordinary complex analysis, the function \( f \) would only be defined on

\[
\mathbb{C} - \{ \text{zeros of } Q \},
\]

but the properties of such a rational function are far clearer when it is considered to be a holomorphic function \( F : S^2 \rightarrow S^2 \), where \( S^2 \) is the Riemann...
sphere, with \( F(z) = \infty \), when \( P(z) = 0 \). In particular, the linear fractional transformations \( T \) defined by

\[
T(z) = \frac{az + b}{cz + d}
\]

where \( a, b, c \) and \( d \) are complex numbers such that \( ad - bc \neq 0 \), extend to holomorphic diffeomorphisms \( T : S^2 \rightarrow S^2 \), the inverse of \( T \) given by

\[
T^{-1}(z) = \frac{dz - b}{-cz + a},
\]

as one verifies by a direct calculation. Moreover, the composition of two linear fractional transformations is another linear fractional transformation, the linear fractional transformations forming a group under composition.

Although the definition of Riemann surface is similar to that of a two-dimensional smooth manifold, the theory of Riemann surfaces is far more rigid. Indeed, we will see later that there are many, many smooth diffeomorphisms of \( S^2 \), while the only holomorphic diffeomorphisms of \( S^2 \) are the linear fractional transformations. To prove the latter fact, we need some results from complex analysis. First note that we can assume without loss of generality that \( F(\infty) = \infty \), since we can arrange this after composition with a linear fractional transformation. But then \( F \) restricts to a holomorphic diffeomorphism of \( \mathbb{C} \), which we also denote by \( F \). It then follows from Theorem 3.3, Chapter 5 of Lang [8] (which is based upon a theorem of Weierstrass regarding essential singularities) that \( F(z) = az + b \), for some complex numbers \( a \) and \( b \). But this is exactly the form of a linear fractional transformation which takes \( \infty \) to \( \infty \).

More details on the theory of Riemann surfaces can be found in many excellent books on this subject, including the highly recommended reference by Forster [5].

We can extend our proof that \( S^2 \) has a smooth manifold to \( n \) dimensions:

**Example 3.** We let

\[
S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : (x_1)^2 + (x_2)^2 + \cdots + (x_{n+1})^2 = 1\},
\]

with the topology it inherits as a subspace of \( \mathbb{R}^{n+1} \), which is automatically Hausdorff and second countable. We can extend the construction of the charts made in Example 2 and perhaps make the computations a little more efficient. Thus we let \( N = (0, 0, \ldots, 1) \) be the north pole in \( \mathbb{R}^{n+1} \), let \( S = (0, 0, \ldots, -1) \) be the south pole, and consider the two open sets

\[
U = S^n - \{N\}, \quad V = S^n - \{S\}.
\]

We claim that there are bijections

\[
\phi : U \rightarrow \mathbb{R}^n, \quad \psi : V \rightarrow \mathbb{R}^n
\]

such that

\[
\mathcal{A} = \{(U, \phi), (V, \psi)\}
\]
is a smooth atlas on $S^n$ which makes it into an $n$-dimensional smooth manifold.

Of course, the construction is the same as in the preceding example, namely we let $\phi$ be the inverse of stereographic projection from the north pole,

$$\phi(x_1, x_2, \ldots, x_{n+1}) = (u^1, u^2, \ldots u^n),$$

where

$$(u^1, u^2, \ldots u^n, 0) = (0, 0, \ldots, 1) + \lambda \left( (x^1, x^2, \ldots, x^{n+1}) - (0, 0, \ldots, 1) \right).$$

Then

$$0 = 1 + \lambda(x^{n+1} - 1) \quad \Rightarrow \quad \lambda = \frac{1}{1-x^{n+1}},$$

so

$$\phi(x_1, x_2, \ldots, x_{n+1}) = u = (u^1, u^2, \ldots u^n) = \frac{1}{1-x^{n+1}}(x^1, x^2, \ldots x^n). \quad (1.7)$$

Thus

$$|u|^2 = u \cdot u = \frac{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2}{(1-x^{n+1})^2} = \frac{1 - (x^{n+1})^2}{(1-x^{n+1})^2} = \frac{1}{1-x^{n+1}}.$$ 

We can now solve for $x^{n+1}$, and obtain

$$|u|^2 + 1 = \frac{2}{1-x^{n+1}}, \quad |u|^2 - 1 = \frac{2x^{n+1}}{1-x^{n+1}}, \quad x^{n+1} = \frac{|u|^2 - 1}{|u|^2 + 1}.$$ 

It follows that

$$(x^1, x^2, \ldots, x^n) = (1-x^{n+1})(u^1, u^2, \ldots u^n) = \frac{2}{|u|^2 + 1}(u^1, u^2, \ldots u^n),$$

and

$$\phi^{-1}(u^1, u^2, \ldots u^n) = (x_1, x_2, \ldots, x_{n+1}) = \frac{1}{|u|^2 + 1}(2u^1, 2u^2, \ldots 2u^n, |u|^2 - 1). \quad (1.8)$$

Formulae (1.7) and (1.8) show that $\phi$ is a homeomorphism from $U$ to $\mathbb{R}^n$.

Similarly, we define $\tilde{\psi} : V \rightarrow \mathbb{R}^n$ to be the inverse of stereographic projection from the south pole,

$$\tilde{\psi}(x_1, x_2, \ldots, x_{n+1}) = (v^1, v^2, \ldots v^n),$$

where

$$(v^1, v^2, \ldots v^n, 0) = (0, 0, \ldots, -1) + \lambda \left( (x^1, x^2, \ldots, x^{n+1}) - (0, 0, \ldots, -1) \right).$$

The only thing that changes is the sign of $x^{n+1}$ in the preceding formulae, so the same argument shows that $\tilde{\psi}$ is a diffeomorphism, and

$$(v^1, v^2, \ldots v^n) = \tilde{\psi}(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{1+x^{n+1}}(x^1, x^2, \ldots x^n).$$
It follows that
\[(x^1, x^2, \ldots, x^n) = (1 - x^{n+1})(u^1, u^2, \ldots, u^n) = (1 + x^{n+1})(v^1, v^2, \ldots, v^n),\]
so
\[\bar{\psi} \circ \phi^{-1}(u^1, u^2, \ldots, u^n) = \frac{1 - x^{n+1}}{1 + x^{n+1}}(u^1, u^2, \ldots, u^n) = \frac{1}{|u|^2}(u^1, u^2, \ldots, u^n). \tag{1.9}\]

Similarly,
\[\phi \circ \bar{\psi}^{-1}(v^1, v^2, \ldots, v^n) = \frac{1}{|v|^2}(v^1, v^2, \ldots, v^n),\]
so we see that \{\((U, \phi), (V, \bar{\psi})\)\} is a smooth atlas on \(S^n\).

As long as we’re not worried about questions of orientation, the atlas we have constructed is perfectly fine, but it turns out that the coordinate change (1.9) which simply inverts a point through the unit sphere in \(\mathbb{R}^n\) reverses orientation (because it reverses the direction of the radial coordinate). We can repair this small defect in our atlas by replacing \(\bar{\psi}\) by \(\psi = R \circ \bar{\psi}\), where
\[R : \mathbb{R}^n \to \mathbb{R}^n \text{ by } R(v^1, \ldots, v^{n-1}, v^n) = (v^1, \ldots, v^{n-1}, -v^n).\]

If we make this small change and specialize to the case \(n = 2\), we recover the same atlas \(A = \{(U, \phi), (V, \psi)\}\) as in Example 2.

We can modify the definition of a smooth manifold slightly to get the definition of an oriented manifold. Suppose that
\[\phi_\alpha = (x^1, \ldots, x^n) : U_\alpha \to \mathbb{R}^n, \quad \phi_\beta = (y^1, \ldots, y^n) : U_\beta \to \mathbb{R}^n\]
are two charts on an \(n\)-dimensional topological manifold \(M\). On the overlap \(\phi_\alpha(U_\alpha \cap U_\beta)\), we can define the Jacobian matrix
\[D(\phi_\beta \circ \phi_\alpha^{-1}) = \begin{pmatrix} \partial y^1/\partial x^1 & \cdots & \partial y^1/\partial x^n \\ \cdots & \cdots & \cdots \\ \partial y^n/\partial x^1 & \cdots & \partial y^n/\partial x^n \end{pmatrix} : \phi_\alpha(U_\alpha \cap U_\beta) \to GL(n, \mathbb{R}),\]
where \(GL(n, \mathbb{R})\) is the space of invertible \(n \times n\) matrices. We say that the charts \((U_\alpha, \phi_\alpha)\) and \((U_\beta, \phi_\beta)\) are coherently oriented if
\[\det(D(\phi_\beta \circ \phi_\alpha^{-1})) : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R} - \{0\}\]
is positive. We say that a smooth atlas \(A = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}\) on \(M\) is oriented if \((U_\alpha, \phi_\alpha)\) and \((U_\beta, \phi_\beta)\) are coherently oriented for every choice of \(\alpha, \beta \in A\). Two oriented smooth atlases \(A\) and \(B\) on \(M\) are said to be equivalent if \(A \cup B\) is an oriented smooth atlas.

**Definition.** An oriented smooth \(n\)-dimensional manifold is an \(n\)-dimensional smooth manifold with an equivalence class of oriented smooth atlases.

Thus for example, the atlas \{\((U, \phi), (V, \psi)\)\} we constructed above on \(S^n\) makes it into an oriented smooth \(n\)-dimensional manifold.
Exercise I. (Due Monday, October 13.) Let $\mathbb{R}P^n$ be the set of all lines which pass through the origin in $\mathbb{R}^{n+1}$. More precisely, we define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{(0,0,\ldots,0)\}$ by

$$(x^1,x^2,\ldots,x^{n+1}) \sim (y^1,y^2,\ldots,y^{n+1})$$

if and only if there exists $\lambda \in \mathbb{R} - \{0\}$ such that

$$(x^1,x^2,\ldots,x^{n+1}) = \lambda(y^1,y^2,\ldots,y^{n+1}),$$

and we let $[x^1,x^2,\ldots,x^{n+1}]$ denote the equivalence class of $(x^1,x^2,\ldots,x^{n+1})$. We can identify $[x^1,x^2,\ldots,x^{n+1}]$ with the line passing through $(0,0,\ldots,0)$ and $(x^1,x^2,\ldots,x^{n+1})$, and thus $\mathbb{R}P^n$ is simply the quotient space

$$\mathbb{R}P^n = \frac{\mathbb{R}^{n+1} - \{(0,0,\ldots,0)\}}{\sim},$$

and we let $\pi : \mathbb{R}^{n+1} - \{(0,0,\ldots,0)\} \to \mathbb{R}P^n$ denote the quotient map. (Alternatively, we can restrict the projection $\pi$ to $S^n \subseteq \mathbb{R}^{n+1} - \{(0,0,\ldots,0)\}$ and regard $\mathbb{R}P^n$ as a quotient of $S^n$ with a projection $\pi : S^n \to \mathbb{R}P^n$ that identifies antipodal points.)

We give $\mathbb{R}P^n$ the quotient topology it inherits from $\mathbb{R}^{n+1} - \{(0,0,\ldots,0)\}$. Thus a subset $U$ lies in $\mathcal{T}_{\mathbb{R}P^n}$ if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} - \{(0,0,\ldots,0)\}$. Show that this topology is Hausdorff and has a countable base. For $1 \leq i \leq n+1$, let

$$U_i = \{[x^1,x^2,\ldots,x^{n+1}] \in \mathbb{R}P^n : x_i \neq 0\}$$

and define $\phi_i : U_i \to \mathbb{R}^n$ by

$$\phi_i([x^1,x^2,\ldots,x^{n+1}]) = \left(\frac{x^1}{x^i}, \frac{x^2}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right)$$

Show that $A = \{(U_i, \phi_i) : 1 \leq i \leq n+1\}$ is a smooth atlas on $\mathbb{R}P^n$ which makes $\mathbb{R}P^n$ into an $n$-dimensional smooth manifold.

Dangerous curve. This atlas we have constructed on $\mathbb{R}P^n$ is not oriented when $n$ is even. In fact, we will see later in the course that there is no oriented atlas on $\mathbb{R}P^n$ when $n$ is even, and in this case we say that $\mathbb{R}P^n$ is a nonorientable smooth manifold of dimension $n$.

Exercise IA. (Do not hand in.) Let $\mathbb{C}P^n$ be the set of all lines which pass through the origin in $\mathbb{C}^{n+1}$. As in the previous exercise, we define an equivalence relation $\sim$ on $\mathbb{C}^{n+1} - \{(0,0,\ldots,0)\}$ by

$$(z^1,z^2,\ldots,z^{n+1}) \sim (w^1,w^2,\ldots,w^{n+1})$$

if and only if there exists $\lambda \in \mathbb{C} - \{0\}$ such that

$$(z^1,z^2,\ldots,z^{n+1}) = \lambda(w^1,w^2,\ldots,w^{n+1}),$$

and we let $[z^1,z^2,\ldots,z^{n+1}]$ denote the equivalence class of $(z^1,z^2,\ldots,z^{n+1})$. We can identify $[z^1,z^2,\ldots,z^{n+1}]$ with the line passing through $(0,0,\ldots,0)$ and $(z^1,z^2,\ldots,z^{n+1})$, and thus $\mathbb{C}P^n$ is simply the quotient space

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1} - \{(0,0,\ldots,0)\}}{\sim},$$

and we let $\pi : \mathbb{C}^{n+1} - \{(0,0,\ldots,0)\} \to \mathbb{C}P^n$ denote the quotient map. (Alternatively, we can restrict the projection $\pi$ to $S^n \subseteq \mathbb{C}^{n+1} - \{(0,0,\ldots,0)\}$ and regard $\mathbb{C}P^n$ as a quotient of $S^n$ with a projection $\pi : S^n \to \mathbb{C}P^n$ that identifies antipodal points.)

We give $\mathbb{C}P^n$ the quotient topology it inherits from $\mathbb{C}^{n+1} - \{(0,0,\ldots,0)\}$. Thus a subset $U$ lies in $\mathcal{T}_{\mathbb{C}P^n}$ if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^{n+1} - \{(0,0,\ldots,0)\}$. Show that this topology is Hausdorff and has a countable base. For $1 \leq i \leq n+1$, let

$$U_i = \{[z^1,z^2,\ldots,z^{n+1}] \in \mathbb{C}P^n : z_i \neq 0\}$$

and define $\phi_i : U_i \to \mathbb{C}^n$ by

$$\phi_i([z^1,z^2,\ldots,z^{n+1}]) = \left(\frac{z^1}{\overline{z^i}}, \frac{z^2}{\overline{z^i}}, \ldots, \frac{z^{i-1}}{\overline{z^i}}, \frac{z^{i+1}}{\overline{z^i}}, \ldots, \frac{z^{n+1}}{\overline{z^i}}\right)$$

Show that $A = \{(U_i, \phi_i) : 1 \leq i \leq n+1\}$ is a smooth atlas on $\mathbb{C}P^n$ which makes $\mathbb{C}P^n$ into an $n$-dimensional smooth manifold.
and we let \([z^1, z^2, \ldots, z^{n+1}]\) denote the equivalence class of \((z^1, z^2, \ldots, z^{n+1})\). We identify \([z^1, z^2, \ldots, z^{n+1}]\) with the line passing through the points \((0,0,\ldots,0)\) and \((z^1, z^2, \ldots, z^{n+1})\), which identifies \(\mathbb{C}P^n\) with the quotient space \(\mathbb{C}P^n = \mathbb{C}^{n+1} - \{(0,0,\ldots,0)\} \sim \).

Show that \(\mathbb{C}P^n\) is a smooth manifold of dimension 2\(n\). Can you show that \(\mathbb{C}P^1\) is diffeomorphic to \(S^2\)?

### 1.3 New manifolds from old

We can construct new manifolds from old by taking products or connected sums. Before describing how to do this, we introduce some terminology. If \(M\) is a smooth manifold with smooth atlas \(A = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}\), we say that a chart \((U, \phi)\) on \(M\) is smooth if \(\phi \circ \phi^{-1}\) and \(\phi_\alpha \circ \phi^{-1}\) are smooth where defined, for all \(\alpha \in A\). If \(p \in M\), there always exists a smooth chart \((U, \phi)\) such that \(p \in U\) and \(\phi(p) = 0\).

There is one really easy way of constructing new manifolds. Note that if \(M\) is a smooth manifold of dimension \(n\) with smooth atlas \(A = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}\) and \(U\) is an open subset of \(M\), we can make \(U\) into a smooth manifold of dimension \(n\) by giving \(U\) the subspace topology and the atlas

\[A \cap U = \{(U_\alpha \cap U, \phi_\alpha | (U_\alpha \cap U)) : \alpha \in A\} \].

Thus any open subset of a smooth manifold is a smooth manifold in its own right.

**Products of manifolds:** Suppose that \(M\) and \(N\) are smooth manifolds of dimensions \(m\) and \(n\) respectively, and suppose that \(M\) and \(N\) have smooth atlases \(A = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}\) and \(B = \{(V_\beta, \phi_\beta) : \beta \in B\}\) respectively. We claim that we can then make the Cartesian product \(M \times N\) into a smooth manifold of dimension \(m + n\).

First, if \(\mathcal{T}_M\) and \(\mathcal{T}_N\) are the topologies on \(M\) and \(N\) respectively, we can set

\[\mathcal{B}_{M \times N} = \{U \times V \subseteq M \times N : U \in \mathcal{T}_M, V \in \mathcal{T}_N\}\).

We then let \(\mathcal{T}_{M \times N}\) be the collection of all unions of elements of \(\mathcal{B}_{M \times N}\). It is straightforward to check that \(\mathcal{T}_{M \times N}\) is a topology on \(M \times N\), and we call it the product topology. It is also not difficult to check that \((M \times N, \mathcal{T}_{M \times N})\) is Hausdorff and has a countable base for its topology.

As a smooth atlas on \(M \times N\), we take the product atlas

\[A \times B = \{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta) : \alpha \in A, \beta \in B\}\].

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Note that
\[ \phi \times \psi : U \times V \longrightarrow \mathbb{R}^{m+n}. \]
It is straightforward to check that the resulting atlas is smooth. Moreover, if \( A \) and \( B \) are oriented smooth atlases, so is the product \( A \times B \).

For example, we could take \( M = S^m \) and \( N = S^n \), each with the atlas described in the previous section. The product atlas on \( S^m \times S^n \) has four elements. We can always enlarge our atlas by including an arbitrary number of new smooth charts without changing the equivalence class of the atlas.

**Quotients:** Let \( \text{Diff}(M) \) denote the set of diffeomorphisms from a smooth manifold \( M \) to itself, a group under composition. Suppose that \( \Gamma \) is a subgroup of \( \text{Diff}(M) \).

We say that \( \Gamma \) acts **freely** on \( M \) if the only \( \gamma \in \Gamma \) which leaves any point of \( M \) fixed is the identity. We say that \( \Gamma \) acts **properly discontinuously** on \( M \) if every \( p \in M \) has an open neighborhood \( U \) such that
\[ \{ \gamma \in \Gamma : \gamma(U) \cap U \neq \emptyset \} \text{ is finite.} \]
If \( \Gamma \) acts freely and properly discontinuously on \( M \), then every \( p \in M \) has an open neighborhood \( U \) such that
\[ \{ \gamma \in \Gamma : \gamma(U) \cap U \neq \emptyset \} = \{ \text{id} \}. \tag{1.10} \]
If \( \Gamma \) acts freely and properly discontinuously on \( M \), we let \( M/\Gamma \) be the space of orbits, the space of equivalence classes of points of \( M \) under the equivalence relation
\[ p \sim q \iff \text{there exists } \gamma \in \Gamma \text{ such that } \gamma(p) = q. \]
We let \( \pi : M \to M/\Gamma \) be the projection which takes \( p \) to the equivalence class \([p]\) of \( p \), and given \( M/\Gamma \) the quotient topology which declares a set \( U \subseteq M/\Gamma \) to be open if and only if \( \pi^{-1}(U) \) is open in \( M \).

**Proposition 1.3.1.** If \( \Gamma \) is a group of diffeomorphisms which acts freely and properly discontinuously on a smooth \( n \)-dimensional manifold \( M \), then \( M/\Gamma \) is a smooth \( n \)-dimensional manifold in such a way that \( \pi : M \to M/\Gamma \) is a smooth local diffeomorphism.

(By saying that \( \pi \) is a **local diffeomorphism**, we mean that any \( p \in M \) has an open neighborhood \( U \) such that \( \pi|U \) is a diffeomorphism.)

Sketch of proof: We leave it to the reader to check that the quotient topology on \( M/\Gamma \) is Hausdorff and second countable. We need to construct the smooth charts on \( M/\Gamma \). For a given choice of \( p \in M \), we let \( (U_p, \phi_p) \) be a smooth chart on \( M \) with \( p \in U_p \) such that \( U_p \) satisfies condition (1.10). Then \( \pi \) maps \( U_p \) homeomorphically onto \( \pi(U_p) \subseteq M/\Gamma \). Hence
\[ (V_p \psi_p) = (\pi(U_p), \phi_p \circ (\pi|U_p)^{-1}) \]
is a chart on $M/\Gamma$. Finally, we check that

$$\{(V_p, \psi_p) : p \in M\}$$

is a smooth atlas on $M/\Gamma$ making it into a smooth $n$-dimensional manifold with the required properties.

For example, suppose that $M = \mathbb{R}^2$ and that $\Gamma$ is the free abelian group generated by the translations

$$\tau_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \tau_1(x, y) = (x + 1, y),$$

$$\tau_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \tau_1(x, y) = (x + u, y + v),$$

where $(u, v) \in \mathbb{R}^2$ and $v > 0$. Then $\Gamma$ is a subgroup of $\text{Diff}(\mathbb{R}^2)$ and $\mathbb{R}^2/\Gamma = T^2$ = torus.

**Connected sums of manifolds:** If $M$ and $N$ are smooth manifolds of dimension $n$, we can construct a new manifold $M \sharp N$ of dimension $n$, called the **connected sum** of $M$ and $N$ by taking disks out of $M$ and $N$ and identifying the results along the boundaries of the disks. If $M$ and $N$ are oriented manifolds, the connected sum will also be oriented.

Here is a more detailed description of the construction: Suppose that $p \in M$ and $q \in N$, and that we have chosen charts

$$(U, \phi) = (U, (x^1, \ldots, x^n)) \quad \text{on} \quad M \quad \text{such that} \quad p \in U \quad \text{and} \quad \phi(p) = 0,$$

and

$$(V, \psi) = (V, (y^1, \ldots, y^n)) \quad \text{on} \quad N \quad \text{such that} \quad q \in V \quad \text{and} \quad \psi(q) = 0.$$ 

We let

$$B_\varepsilon(0) = \{x \in \mathbb{R}^n : |x| < \varepsilon\}, \quad \bar{B}_\varepsilon(0) = \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}$$

and suppose that

$$\bar{B}_2(0) \subset \phi(U) \quad \text{and} \quad \bar{B}_2(0) \subset \psi(V),$$

by rescaling the coordinates if necessary. We suppose that $M$ and $N$ have atlases

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in \mathcal{A}\} \cup \{(U, \phi)\}, \quad \mathcal{B} = \{(V_\beta, \phi_\beta) : \beta \in \mathcal{B}\} \cup \{(V, \psi)\}$$

such that for each $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B},$

$$U_\alpha \cap \phi^{-1}(\bar{B}_{1/2}(0)) = \emptyset, \quad V_\beta \cap \psi^{-1}(\bar{B}_{1/2}(0)) = \emptyset$$

and

$$\bigcup\{U_\alpha : \alpha \in \mathcal{A}\} \cup \phi^{-1}(\bar{B}_2(0)) = m, \quad \bigcup\{V_\beta : \beta \in \mathcal{B}\} \cup \phi^{-1}(\bar{B}_2(0)) = N.$$
We then define
\[ \eta : B_2(0) - \bar{B}_{1/2}(0) \rightarrow B_2(0) - \bar{B}_{1/2}(0) \quad \text{by} \quad \eta(x) = \frac{x}{|x|^2}. \]

Note that \( \eta \) is simply an inversion in the sphere
\[ S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}, \]
which is a diffeomorphism of \( B_2(0) - \bar{B}_{1/2}(0) \). Although it is orientation-reversing, if we let
\[ R : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad \text{by} \quad R(x^1, \ldots, x^{n-1}, x^n) = (x^1, \ldots, x^{n-1}, -x^n), \]
then \( R \circ \eta \) will be orientation-preserving. We set
\[ \hat{M} = M - \phi^{-1}(\bar{B}_{1/2}(0)), \quad \hat{N} = N - \psi^{-1}(\bar{B}_{1/2}(0)), \]
and define an equivalence relation \( \sim \) on the disjoint union of \( \hat{M} \) and \( \hat{N} \) by decreeing that \( r \in \phi^{-1}(\bar{B}_2(0)) \cap \hat{M} \) is equivalent to \( s \in \psi^{-1}(\bar{B}_2(0)) \cap \hat{N} \) (which we write as \( r \sim s \)) if and only if
\[ \phi(r) = R \circ \eta \circ \psi(s). \]

Finally, we denote the set of equivalence classes by \( M \sharp N \).

There is a unique topology on \( M \sharp N \) which makes the inclusions
\[ \hat{M} \subseteq M \sharp N \quad \text{and} \quad \hat{N} \subseteq M \sharp N \]
homeomorphisms onto open subsets. One checks once again that this topology is Hausdorff and second countable. Note that in \( M \sharp N \), \( \phi^{-1}(\bar{B}_2(0)) \cap \hat{M} \) and \( \psi^{-1}(\bar{B}_2(0)) \cap \hat{N} \) are identified to a set \( W \) which is open for this topology, and we can define
\[ \chi : W \rightarrow \mathbb{R}^n \quad \text{by} \quad \chi([r]) = \phi(r), \]
where \([r]\) is the equivalence class of \( r \in U \cap \hat{M} \). Finally, we give \( M \sharp N \) the atlas
\[ \{(U_\alpha, \phi_\alpha) : \alpha \in A\} \cup \{(V_\beta, R \circ \phi_\beta) : \beta \in B\} \cup \{(W, \chi)\}, \]
and check that it is smooth. This gives \( M \sharp N \) the unique smooth manifold structure such that the inclusions
\[ \hat{M} \subseteq M \sharp N \quad \text{and} \quad \hat{N} \subseteq M \sharp N \]
are diffeomorphisms onto open subsets of \( M \sharp N \), and finishes our construction of \( M \sharp N \).

Note that the connected sum of two oriented manifolds of dimension \( n \) is also oriented. The notions of product and connected sum are sufficient to enable us to describe the well-known classification of compact path connected surfaces. We first recall the definitions from topology, a subject treated in [15]:
**Definition.** A topological space \((X, T)\) is **compact** if every open cover of \(X\) has a finite subcover.

The Heine-Borel theorem from real analysis states that a subspace of \(\mathbb{R}^n\) with the subspace topology is compact if and only if it is closed and bounded.

**Definition.** A topological space \((X, T)\) is said to be **path connected** if whenever \(p\) and \(q\) are points of \(X\), there is a continuous map

\[ \gamma : [0, 1] \to X \quad \text{such that} \quad \gamma(0) = p \quad \text{and} \quad \gamma(1) = q. \]

More generally, we say that \((X, T)\) is **connected** if the only subsets of \(X\) which are both open and closed are \(\emptyset\) and \(X\) itself. In topology courses, it is shown that path connected spaces are also connected, and for topological manifolds, it can be shown that connectedness is equivalent to path connectedness.

It can be proven that any one-dimensional path connected smooth manifold is diffeomorphic to either \(\mathbb{R}\) or \(S^1\); a proof of this fact is given in the appendix to a highly recommended short book by Milnor [12].

There is also a simple classification of oriented compact path connected two-dimensional manifolds up to diffeomorphism in terms of the genus \(g \in \{0\} \cup \infty\). We start with the sphere \(S^2\), which is said to have genus zero and the torus \(S^1 \times S^1\) which has genus one. There is then a unique oriented compact connected surface of each genus \(g\) when \(g > 1\), which is the connected sum of \(g\) copies of the torus \(S^1 \times S^1\).

There is also a series of nonorientable compact path connected two-dimensional manifolds, the connected sum of \(h\) copies of \(\mathbb{R}P^2\) where \(h \geq 1\).

**Definition.** A topological space \((X, T)\) is **simply connected** if it is path connected, and if every continuous loop

\[ \gamma : [0, 1] \to M, \quad \text{with} \quad \gamma(0) = p = \gamma(1) \]

can be continuously contracted to a point. That means that there is a continuous map \(H : [0, 1] \times [0, 1] \to M\) such that

\[ H(t, 0) = H(0, u) = H(1, u) = p, \quad H(t, 1) = \gamma(t). \]

(We can think of \(H\) as defining a family of loops \(\{\gamma_u\}\) at \(p\) by \(\gamma_u(t) = H(t, u)\) such that \(\gamma_0\) is a constant and \(\gamma_1 = \gamma\).)

The only compact simply connected two-dimensional manifold is \(S^2\).

One can ask for classification of higher-dimensional manifolds. Work on these topics forms the subject of more advanced courses in geometry and topology.

One of the first problems considered by Poincaré in the development of what he called “analysis situs” concerned the classification of three-dimensional manifolds. Indeed, Poincaré raised the following question in 1904 which came to be known as the Poincaré conjecture: Is any compact simply connected three-dimensional topological manifold homeomorphic to \(S^3\)? During the twentieth
century this was regarded as one of the most important open problems in mathematics, and it became one of the seven Clay Mathematics Institute Millennium Prize problems for which a million dollar prize was offered in 2000. The Poincaré conjecture was settled by Grigory Perelman in 2002-03. Perelman went on to prove Thurston’s geometrization conjecture, which provides a classification for three-dimensional manifolds in terms of geometric structures on those manifolds. Crucial to the solution were techniques for studying a nonlinear version of the heat equation, a nonlinear partial differential equation applied to Riemannian metrics on three-manifolds which studied the evolution of the metric under “Ricci flow,” as developed by Richard Hamilton. (An introductory survey on Perelman’s work can be found in [1].)

It turns out that every topological manifold of dimension less than or equal to three has a unique smooth manifold structure, so the classification of topological manifolds of dimension \( \leq 3 \) is the same as that for smooth manifolds.

In contrast, classification of four-dimensional manifolds is still in a fragmented state. An additional complication is that although very three-dimensional topological manifold has a unique smooth manifold structure, in dimension four there is a divergence between the topological and smooth classifications. For example, breakthroughs of Freedman and Donaldson showed that there are uncountably many ways of making \( \mathbb{R}^4 \) into a smooth manifold—this is the only dimension in which this can happen. Many examples are known of compact simply connected four-dimensional topological manifolds which have no smooth structures, and others which have infinitely many. (On the other hand, it is not known whether the smooth four-dimensional Poincaré conjecture is true: this conjecture states that the topological four-sphere \( S^4 \) has only one smooth structure.) Once again, partial differential equations on manifolds provide an essential technique for proving these results, this time the Yang-Mills and Seiberg-Witten equations that come from mathematical physics.

These results reverse the usual direction of application. Ordinarily one would think of topology and geometry as forming a foundation for partial differential equations and mathematical physics. It is remarkable that many of the most important ideas within the topology of low-dimensional manifolds obtained in recent years have gone in the reverse direction.

The reader can find many further examples of smooth manifolds in Chapter 1 of Lee [10].

1.4 The tangent space

Our next goal is to define the various concepts from several variable calculus in a manner which does not make explicit use of local coordinates, insofar as possible. We start off with differentiation, with the notion of directional derivative at a point, which is encapsulated in the definition of tangent vector.

A tangent vector might represent the instantaneous velocity of a particle moving in a smooth manifold. The problem is that once we leave the comfort of Euclidean space, there is no preferred coordinate system in which to measure
the components of a vector—all coordinate systems are on the same footing. Tangent vectors must be defined so that their components can be allowed to change from coordinate system to coordinate system.

Suppose that \( M \) is a smooth manifold and \( p \in M \), and that \( \mathcal{F}(p) \) denotes the space of pairs \((U, f)\) where \( U \) is an open subset of \( M \) containing \( p \) and \( f : U \to \mathbb{R} \) is a smooth real-valued function. For simplicity of notation, we will usually leave out the open set \( U \) and simply write \( f \) for a typical element of \( \mathcal{F}(p) \). We can define an equivalence relation \( \sim \) on \( \mathcal{F}(p) \) by

\[
f \sim g \iff f \equiv g \text{ on some open neighborhood of } p.
\]

The equivalence class of an element \( f \in \mathcal{F}(p) \) is then called the \textit{germ} of \( f \) at \( p \).

**Definition.** A tangent vector at \( p \in M \) is a function \( v : \mathcal{F}(p) \to \mathbb{R} \) which satisfies the following axioms:

1. \( v(f) = v(g) \) if \( f \equiv g \) on some open neighborhood of \( p \), that is, \( v(f) \) depends only on the germ of \( f \) at \( p \).
2. \( v(cf + g) = cv(f) + v(g) \), for \( f, g \in \mathcal{F}(p) \) and \( c \in \mathbb{R} \).
3. \( v(fg) = f(p)v(g) + g(p)v(f) \), for \( f, g \in \mathcal{F}(p) \).

We let \( T_p M \) denote the collection of tangent vectors to \( M \) at \( p \) and call it the tangent space to \( M \) at \( p \).

In axioms 2 and 3, if \( f \) is defined on \( U \) while \( g \) is defined on \( V \), we regard \( cf + g \) and \( fg \) as defined on \( U \cap V \). The third axiom is known as the Leibniz rule for differentiation. It is easily verified that \( T_p M \) is a real vector space.

**Examples.** Suppose that \( M \) is an \( n \)-dimensional smooth manifold, \( p \in M \), and that

\[
\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n
\]

is a smooth coordinate system on \( M \) with \( p \in U \). If \( f \in \mathcal{F}(p) \), we then define

\[
\left. \frac{\partial}{\partial x^i} \right|_p (f) = D_i(f \circ \phi^{-1})(\phi(p)) \in \mathbb{R},
\]

where \( D_i \) denotes differentiation with respect to the \( i \)-th coordinate direction in the Euclidean space \( \mathbb{R}^n \). The local coordinates therefore provide a collection of “directional derivative operators” which clearly depends only on the germ of \( f \) at \( p \),

\[
f \equiv g \text{ on some neighborhood of } p \implies \left. \frac{\partial}{\partial x^i} \right|_p (f) = \left. \frac{\partial}{\partial x^i} \right|_p (g).
\]

One readily verifies that each of these directional derivative operators satisfies the other two axioms, and the Leibniz rule,

\[
\left. \frac{\partial}{\partial x^i} \right|_p (fg) = \left( \left. \frac{\partial}{\partial x^i} \right|_p (f) \right) g(p) + f(p) \left( \left. \frac{\partial}{\partial x^i} \right|_p (g) \right).
\]
reflects a familiar property of differentiation in Euclidean space. Hence each directional derivative operator
\[ \frac{\partial}{\partial x^i} \bigg|_p \]
is an element of the tangent space \( T_p M \). Moreover, each linear combination
\[ \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i} \bigg|_p \]
lies within the tangent space \( T_p M \). We will see shortly that these are the only elements of \( T_p M \), so that
\[ \left( \frac{\partial}{\partial x^1} \bigg|_p, \ldots, \frac{\partial}{\partial x^n} \bigg|_p \right) \]
is a basis for the tangent space \( T_p M \). Indeed, this follows from:

**Theorem 1.4.1.** Suppose that \( M \) is a smooth \( n \)-dimensional manifold, that \( p \in M \) and that \((U, \phi) = (U, (x^1, \ldots, x^n))\) is a smooth chart on \( M \) with \( p \in U \). If \( v \in T_p M \), then
\[ v = \sum_{i=1}^{n} v(x^i) \frac{\partial}{\partial x^i} \bigg|_p. \] (1.11)

To prove this, we first make the simplifying assumption that \( \phi(p) = 0 \) and that \( V = \phi(U) \) is a convex subset of \( \mathbb{R}^n \). Our proof is based upon a series of lemmas.

**Lemma 1.** If \( v \in T_p M \) and \( c \) is a constant function, then
\[ v(c) = 0. \]

By axiom 2 for tangent vectors,
\[ v(1) = v(1 \cdot 1) = 1v(1) + 1v(1), \text{ so } v(1) = 2v(1) \text{ or } v(1) = 0. \]

It then follows from axiom 1 that \( v(c) = cv(1) = 0 \). QED

Before stating the next lemma, we agree to let \((\bar{x}^1, \ldots, \bar{x}^n)\) denote the standard Euclidean coordinate system on \( \mathbb{R}^n \).

**Lemma 2.** If \( V \) is a convex neighborhood of \( 0 \in \mathbb{R}^n \) and \( \bar{f} : V \to \mathbb{R} \) is a \( C^\infty \) function, then there are \( C^\infty \) functions \( \bar{g}_1, \ldots, \bar{g}_n : V \to \mathbb{R} \) such that
\[ 1. \quad \bar{f}(\bar{x}^1, \ldots, \bar{x}^n) = \bar{f}(0) + \sum_{i=1}^{n} \bar{x}^i \bar{g}_i, \text{ and} \]
\[ 2. \quad \bar{g}_i(0) = (D_i \bar{f})(0). \]

Indeed, it follows from the fundamental theorem of calculus and the chain rule.
that
\[
\tilde{f}(\bar{x}^1, \ldots, \bar{x}^n) - \tilde{f}(0, \ldots, 0) = \int_0^1 \frac{d}{dt} [\tilde{f}(t\bar{x}^1, \ldots, t\bar{x}^n)] \, dt = \sum_{i=1}^n \bar{x}^i \tilde{g}_i(\bar{x}^1, \ldots, \bar{x}^n),
\]
where
\[
\tilde{g}_i(\bar{x}^1, \ldots, \bar{x}^n) = \int_0^1 D_i \tilde{f}(t\bar{x}^1, \ldots, t\bar{x}^n) \, dt,
\]
which is clearly a $C^\infty$ function on $V$. Moreover, $\tilde{g}_i$ clearly satisfies the two conditions of the Lemma. QED

**Lemma 3.** If $(U, \phi) = (U, (x^1, \ldots, x^n))$ is a smooth chart on $M$ with $p \in U$, $\phi(p) = 0$ and $V = \phi(U)$ a convex neighborhood of $0$, and $f : U \to \mathbb{R}$ is a $C^\infty$ function, then there are $C^\infty$ functions $g_1, \ldots, g_n : U \to \mathbb{R}$ such that
\[
f = f(p) + \sum_{i=1}^n x^i g_i \quad \text{and} \quad g_i(p) = \frac{\partial}{\partial x^i} \bigg|_p (f).
\]

To prove this, we simply pull Lemma 2 back to $U$. QED

Proof of the theorem: If $v \in T_p M$ and $f \in \mathcal{F}(p)$, it follows from Lemma 3 and 1 that
\[
v(f) = v \left( f(p) + \sum_{i=1}^n x^i g_i \right) = \sum_{i=1}^n v(x^i) g_i(p) + \sum_{i=1}^n x^i(p) v(g_i) = \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \bigg|_p (f),
\]
where we have used the assumption that $x^i(p) = 0$.

Finally, we note that the assumptions that $V$ be convex and that $x^i(p) = 0$ are harmless. The first can always be arranged by contracting $U$ and the second by subtracting constants from the coordinates; that is, if $x^i(p) = a^i$, we can set $y^i = x^i - a^i$, and note that
\[
v(x^i) = v(y^i) \quad \text{and} \quad \frac{\partial}{\partial x^i} \bigg|_p = \frac{\partial}{\partial y^i} \bigg|_p . \quad \text{QED}
\]

To simplify notation henceforth, we will write
\[
\frac{\partial f}{\partial x^i}(p) \quad \text{for} \quad \frac{\partial}{\partial x^i} \bigg|_p (f).
\]
Corollary 1.4.2. Suppose that $M$ is a smooth $n$-dimensional manifold and that $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ are two smooth coordinate systems defined on neighborhoods of $p \in M$. Then

$$\frac{\partial}{\partial y^j}\bigg|_p = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^j}(p) \frac{\partial}{\partial x^i}\bigg|_p \quad \text{and} \quad v(x^i) = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^j}(p)v(y^j),$$

(1.12)

for $v \in T_pM$.

For the first of these, we apply (1.11) with

$$v = \frac{\partial}{\partial y^j}\bigg|_p$$

to obtain

$$\frac{\partial}{\partial y^j}\bigg|_p = \sum_{i=1}^{n} \frac{\partial}{\partial y^j} \bigg|_p \left( x^i \right) \frac{\partial}{\partial x^i}\bigg|_p ;$$

the second equation follows from (1.11) and the first.

The first of formulae (1.12) states that the basis vectors for $T_pM$ transform under change of coordinates via the “chain rule,” while the second describes how the components of these vectors transform.

1.5 The cotangent space

We now encounter the first subtlety in our efforts to extend calculus from Euclidean space to arbitrary smooth manifolds—vectors come in two types, tangent vectors and cotangent vectors. In the language of physics, as explained in Misner, Thorne and Wheeler’s exposition of relativity [14], for example, these are called contravariant and covariant vectors, respectively.

Definition. A cotangent vector at $p \in M$ is a linear functional

$$\alpha : T_pM \rightarrow \mathbb{R}.$$

We let $T^*_pM$ denote the collection of cotangent vectors to $M$ at $p$ and call it the cotangent space to $M$ at $p$.

Of course, in the language of linear algebra, $T^*_pM$ is simply the dual space to $T_pM$.

For a simple example of a cotangent vector, we can consider the differential of a smooth function $f : M \rightarrow \mathbb{R}$ at $p$. By definition, this is the element

$$df|_p \in T^*_pM \quad \text{defined by} \quad df|_p(v) = v(f).$$

Note that

$$dx^j|_p \left( \frac{\partial}{\partial x^i}\bigg|_p \right) = \frac{\partial}{\partial x^i}\bigg|_p (x^j) = \frac{\partial x^j}{\partial x^i}(p) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$
and hence \((dx^1|_p, \ldots, dx^n|_p)\) is the basis for \(T^*_p M\) which is dual to 
\[
\left( \frac{\partial}{\partial x^1} \bigg|_p, \ldots, \frac{\partial}{\partial x^n} \bigg|_p \right).
\]

**Proposition 1.5.1.** If \(U\) is an open neighborhood of \(p \in M\) and \(f : U \to \mathbb{R}\) is a smooth function, then 
\[
df|_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) dx^i|_p.
\]

Indeed, it follows the definition of differential and from Theorem 1.4.1 that 
\[
df|_p(v) = v(f) = \sum_{i=1}^{n} v(x^i) \frac{\partial}{\partial x^i}\bigg|_p (f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) dx^i|_p(v).
\]

**Corollary 1.5.2.** Suppose that \(M\) is a smooth \(n\)-dimensional manifold and that \((x^1, \ldots, x^n)\) and \((y^1, \ldots, y^n)\) are two smooth coordinate systems defined on neighborhoods of \(p \in M\). Then 
\[
dx^i|_p = \sum_{j=1}^{n} \frac{\partial x^i}{\partial y^j}(p) dy^j|_p \quad \text{and} \quad \alpha \left( \frac{\partial}{\partial y^j} \bigg|_p \right) = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^j}(p) \alpha \left( \frac{\partial}{\partial x^i} \bigg|_p \right).
\]  \hspace{1cm} (1.13)

if \(\alpha \in T^*_p M\).

The proof is immediate. Notice that our notation makes the transformation formulae (1.12) and (1.13) almost impossible to get wrong. They are both expressions of the chain rule. In the language of classical tensor analysis, (1.12) and (1.13) describe how the components of contravariant and covariant vectors transform under change of coordinates. This terminology has stuck, although it is exactly backwards, as we next explain.

Suppose that \(M\) and \(N\) are smooth manifolds of dimensions \(m\) and \(n\) respectively, and that \(F : M \to N\) is a smooth map. If \(p \in M\), we can define 
\[
(F_*)_p : T_p M \longrightarrow T_{F(p)} N \quad \text{by} \quad (F_*)_p(v)(f) = v(f \circ F),
\]
whenever \(f\) is a smooth real-valued function defined on some neighborhood of \(F(p) \in N\). We easily check that this formula for \((F_*)_p(v)\) does indeed define an element of \(T_{F(p)} N\) and that \((F_*)_p\) is a linear map. Indeed, we will think of \((F_*)_p\) as a “linear approximation” to \(F\) near \(p\). However, note that the linear map \((F_*)_p\) points in the same direction, which is characteristic of what is called a covariant functor,
\[
(F : M \to N) \quad \mapsto \quad ((F_*)_p : T_p M \to T_{F(p)} N),
\]

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although we have called $T_p M$ the space of contravariant vectors.

If $\phi = (x^1, \ldots, x^n)$ and $\psi = (y^1, \ldots, y^m)$ are smooth charts defined on neighborhoods of $p \in M$ and $q = F(p) \in N$ respectively, then it follows from Theorem 1.4.1 that

\[
(F_*)_p \left( \frac{\partial}{\partial x^i} \bigg|_p \right) = \sum_{j=1}^m (F_*)_p \left( \frac{\partial}{\partial x^i} \bigg|_p \right) (y^j) \frac{\partial}{\partial y^j} \bigg|_{F(p)}
\]

\[
= \sum_{j=1}^m \frac{\partial}{\partial x^i} \bigg|_p (y^j \circ F) \frac{\partial}{\partial y^j} \bigg|_{F(p)} = \sum_{j=1}^m \frac{\partial(y^j \circ F)}{\partial x^i}(p) \frac{\partial}{\partial y^j} \bigg|_{F(p)}.
\]

In other words, if

\[
v = \begin{pmatrix} \frac{\partial}{\partial x^1} \bigg|_p & \cdots & \frac{\partial}{\partial x^n} \bigg|_p \end{pmatrix} \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{pmatrix} \in T_p M,
\]

where $\dot{x}^1, \ldots, \dot{x}^n$ are the components of $v$, then

\[
(F_*)_p(v) = \begin{pmatrix} \frac{\partial}{\partial y^1} \big|_{F(p)} & \cdots & \frac{\partial}{\partial y^m} \big|_{F(p)} \end{pmatrix} \begin{pmatrix} \frac{\partial(y^1 \circ F)}{\partial x^1} & \cdots & \frac{\partial(y^1 \circ F)}{\partial x^n} \\ \vdots & \cdots & \vdots \\ \frac{\partial(y^m \circ F)}{\partial x^1} & \cdots & \frac{\partial(y^m \circ F)}{\partial x^n} \end{pmatrix} (p) \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{pmatrix} \in T_q N.
\]

We can therefore say that $(F_*)_p$ is represented by the Jacobian matrix

\[
D(\psi \circ F \circ \phi^{-1})(\phi(p)) = \begin{pmatrix} \frac{\partial(y^1 \circ F)}{\partial x^1} & \cdots & \frac{\partial(y^1 \circ F)}{\partial x^n} \\ \vdots & \cdots & \vdots \\ \frac{\partial(y^m \circ F)}{\partial x^1} & \cdots & \frac{\partial(y^m \circ F)}{\partial x^n} \end{pmatrix} (p).
\]

We can think of the coordinates $(\dot{x}^1, \ldots, \dot{x}^n)$ on $T_p M$ as approximations to the coordinates $(x^1, \ldots, x^n)$ on $M$ and $(F_*)_p$ as the “linearization” of $F$ at $p$ represented in coordinates by matrix multiplication

\[
\begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{pmatrix} \mapsto \begin{pmatrix} \dot{y}^1 \\ \vdots \\ \dot{y}^m \end{pmatrix} = \begin{pmatrix} \frac{\partial(y^1 \circ F)}{\partial x^1} & \cdots & \frac{\partial(y^1 \circ F)}{\partial x^n} \\ \vdots & \cdots & \vdots \\ \frac{\partial(y^m \circ F)}{\partial x^1} & \cdots & \frac{\partial(y^m \circ F)}{\partial x^n} \end{pmatrix} (p) \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{pmatrix}.
\]

Similarly, we can define a linear map on cotangent spaces,

\[
(F^*)_p : T^*_q N \to T^*_p M \quad \text{by} \quad (F^*)_p(\alpha)(v) = \alpha((F_*)_p(v)),
\]

which goes the opposite direction, characteristic of a contravariant functor,

\[
(F : M \to N) \mapsto ((F^*)_p : T^*_F(p) N \to T^*_p M),
\]

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although $T^*_pM$ is called the space of covariant vectors. This time, we find that if
\[
\alpha = \left( \alpha_1 \cdots \alpha_m \right)
\begin{pmatrix}
\left( dy^1 \right)_{F(p)} \\
\vdots \\
\left( dy^m \right)_{F(p)}
\end{pmatrix}
\in T^*_pF(p)N,
\]
then
\[
\left( F^* \right)_p(\alpha) = \left( \alpha_1 \cdots \alpha_m \right)
\begin{pmatrix}
\frac{\partial(y^1 \circ F)}{\partial x^1} \\
\vdots \\
\frac{\partial(y^m \circ F)}{\partial x^1}
\end{pmatrix}
(p)
\begin{pmatrix}
\left( dx^1 \right)_p \\
\vdots \\
\left( dx^m \right)_p
\end{pmatrix}
\in T^*_pM.
\]

An important special case occurs when $M = (a, b)$, an open interval of real numbers. We let $t$ denote the coordinate on $(a, b) \subseteq \mathbb{R}$ and let
\[
\left. \frac{d}{dt} \right|_{t_0} \in T_{t_0}(a, b) = T_{t_0}\mathbb{R}
\]
be the standard basis element. Suppose that $M$ is an $n$-dimensional smooth manifold and that $\gamma : (a, b) \rightarrow M$ is a smooth curve. Then, whenever $f$ is a smooth real-valued function defined on some neighborhood of $\gamma(t_0)$ in $M$,
\[
\left( \gamma *_{t_0} \frac{d}{dt} \right|_{t_0} (f) = \left. \frac{d}{dt} \right|_{t_0} (f \circ \gamma) = (f \circ \gamma)'(t).
\]
It follows from the chain rule that in terms of local coordinates $(x^1, \ldots, x^n)$,
\[
\left. \frac{d}{dt} \right|_{t_0} \left( \gamma * \right)_{t_0} = \sum_{i=1}^{n} \left( \gamma * \right)_{t_0} \left. \frac{d}{dt} \right|_{t_0} (x^i) \frac{\partial}{\partial x^i} \gamma(t_0) = \sum_{i=1}^{n} (x^i \circ \gamma)'(t_0) \frac{\partial}{\partial x^i} \gamma(t_0).
\]
We will use the notation
\[
\gamma'(t_0) = \left( \gamma * \right)_{t_0} \left( \frac{d}{dt} \right|_{t_0}
\]
and call $\gamma'(t_0)$ the velocity vector to $\gamma$ at $t_0$. We can then regard $T_pM$ as consisting of all possible velocity vectors which pass through $p$ at some time $t_0$.

**Proposition 1.5.3.** If $M$, $N$ and $P$ are smooth manifolds, and $F : M \rightarrow N$ and $G : N \rightarrow P$ are smooth maps, then
\[
((G \circ F)_*)_p = (G_*)_{F(p)} \circ (F_*)_p, \quad (G \circ F)^*_p = F^*_p \circ G^*_p_{F(p)}.
\]

The proof of the first of these equations is
\[
((G \circ F)_*)_p(v)(f) = v(f \circ G \circ F) = (F_*)_p(v)(f \circ G) = (G_*)_{F(p)} \circ (F_*)_p(v)(f),
\]

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and the proof of the second is similar.

In particular, if \( \gamma : (a, b) \to M \) is a smooth curve with \( \gamma(t_0) = p \) and \( F : M \to N \) is a smooth map, then

\[
(F_*)_p(\gamma'(t_0)) = (F \circ \gamma)'(t_0).
\]

**Proposition 1.5.4.** Suppose that \( F : M \to N \) is a smooth map, that \( p \in M \) and that \( V \) is an open neighborhood of \( F(p) \) in \( N \). If \( f : V \to \mathbb{R} \) is a smooth functions, then

\[
(F_*)_p(df|_{F(p)}) = d(f \circ F)|_p
\]  

(1.15)

Proof:

\[
(F_*)_p(df|_{F(p)})(v) = df|_{F(p)}((F_*)_p(v)) = (F_*)_p(v)(f) = (v)(f \circ F) = d(f \circ F)|_p(v).
\]

### 1.6 The tangent and cotangent bundles

Suppose that \( M \) is a smooth \( n \)-dimensional manifold with atlas \( \mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in \mathcal{A} \} \). We can then take the disjoint union of all the tangent spaces to \( M \),

\[
TM = \coprod \{ T_pM : p \in M \} \quad \text{(disjoint union),}
\]

and define a projection \( \pi : TM \to M \) by \( \pi(T_pM) = \{p\} \). We call this union the tangent bundle of \( M \).

Suppose that \( (U_\alpha, \phi_\alpha) \) is one of the charts in \( \mathcal{A} \), with \( \phi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha) \). We let \( \tilde{U}_\alpha = \pi^{-1}(U_\alpha) \) and define “velocity coordinates”

\[
\tilde{x}_i^\alpha : \tilde{U}_\alpha \to \mathbb{R} \quad \text{by} \quad \tilde{x}_i^\alpha \left( \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j^\alpha}|_p \right) = a^i.
\]

We can then define a map

\[
\tilde{\phi}_\alpha : \tilde{U}_\alpha \to \mathbb{R}^{2n} \quad \text{by} \quad \tilde{\phi}_\alpha = (x_1^\alpha \circ \pi, \ldots, x_n^\alpha \circ \pi, \tilde{x}_1^\alpha, \ldots, \tilde{x}_n^\alpha).
\]

Note that on the overlap \( \tilde{U}_\alpha \cap \tilde{U}_\beta \),

\[
\tilde{x}_i^\alpha \left( \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j^\beta}|_p \right) = \tilde{x}_i^\alpha \left( \sum_{j,k=1}^{n} b_j \frac{\partial x_k^\alpha}{\partial x_j^\beta}|_p \frac{\partial}{\partial x_k^\alpha}|_p \right) = \sum_{j=1}^{n} \frac{\partial x_k^\alpha}{\partial x_j^\beta}(p) b_j = \sum_{j=1}^{n} \frac{\partial x_k^\alpha}{\partial x_j^\beta}(p) \tilde{x}_j^\beta \left( \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j^\beta}|_p \right)
\]
so
\[ \dot{x}^i_\alpha = \sum_{j=1}^{n} \left( \frac{\partial x^j_\beta}{\partial x^i_\alpha} \circ \pi \right) \dot{x}_\beta^j. \]

It follows that the components of
\[ \tilde{\phi}_\beta = (x^1_\beta \circ \pi, \ldots, x^n_\beta \circ \pi, \dot{x}^1_\beta, \ldots, \dot{x}^n_\beta) \]
depend smoothly on the components of \( \tilde{\phi}_\alpha \), or equivalently,
\[ \tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} : \tilde{\phi}_\alpha(U_\alpha \cap \tilde{U}_\beta) \longrightarrow \tilde{\phi}_\beta(U_\alpha \cap \tilde{U}_\beta) \]
is smooth.

To make \( TM \) into a smooth manifold we need to give it a topology. We let
\[ B = \{ V \subseteq TM : \tilde{\phi}_\alpha(V) \text{ is open in } \mathbb{R}^{2n} \}, \]
and decree that the open subsets of \( TM \) will be the union of elements of \( B \). It is easy to check that is indeed a topology such that each \( \tilde{\phi}_\alpha \) is a homeomorphism from \( U_\alpha \) onto an open subset of \( \mathbb{R}^{2n} \). Moreover, one can check that this topology is Hausdorff and has a countable base. Thus \( A = \{ (U_\alpha, \tilde{\phi}_\alpha) : \alpha \in A \} \) is a smooth atlas on \( TM \) which makes \( TM \) into a smooth manifold of dimension \( 2n \).

Similarly, we can let
\[ T^*M = \coprod \{ T^*_p M : p \in M \} \quad \text{(disjoint union)}, \]
and define a projection \( \pi : T^*M \to M \) by \( \pi(T^*_p M) = \{ p \} \). We call \( T^*M \) the cotangent bundle of \( M \). We can then construct coordinates just like we did for the tangent bundle.

Thus if \( (U_\alpha, \phi_\alpha) \) is one of the charts in \( A \), with \( \phi_\alpha = (x^1_\alpha, \ldots, x^n_\alpha) \), we let \( \tilde{U}_\alpha = \pi^{-1}(U_\alpha) \) and define “momentum coordinates”
\[ p^1_\alpha, \ldots, p^n_\alpha : \tilde{U}_\alpha \to \mathbb{R} \quad \text{by} \quad p^i_\alpha = \left( \sum_{j=1}^{n} a_j dx^j_\alpha \right) |_p = a_i. \]

We can then define
\[ \tilde{\phi}_\alpha : \tilde{U}_\alpha \to \mathbb{R}^{2n} \quad \text{by} \quad \tilde{\phi}_\alpha = (x^1_\alpha \circ \pi, \ldots, x^n_\alpha \circ \pi, p^1_\alpha, \ldots, p^n_\alpha). \]

This time, we have
\[ p^i_\alpha = \sum_{j=1}^{n} \left( \frac{\partial x^j_\beta}{\partial x^i_\alpha} \circ \pi \right) p^j_\beta, \]
but it is still the case that \( \tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} \) is smooth on overlaps. Just as before, we can make \( T^*M \) into a smooth manifold of dimension \( 2n \).

**Definition.** If \( U \) is an open subset of \( M \), a smooth vector field on \( U \) is a smooth map
\[ X : U \longrightarrow TM \quad \text{such that} \quad \pi \circ X = \text{id}_U. \]
**Definition.** If $U$ is an open subset of $M$, a *smooth one-form* on $M$ is a smooth map 
\[ \omega : U \longrightarrow T^*M \] such that \( \pi \circ \omega = \text{id}_U \).

Suppose that \((U, \phi) = (U, (x^1, \ldots, x^n))\) is a smooth coordinate system on $M$. Then we can define smooth vector fields
\[ \frac{\partial}{\partial x^i} : U \longrightarrow TM \] by
\[ \frac{\partial}{\partial x^i} \bigg|_p = \frac{\partial}{\partial x^i} \bigg|_p, \]
for \( 1 \leq i \leq n \). More generally, if \( f^1, \ldots, f^n : U \longrightarrow \mathbb{R} \) are smooth functions on $U$, we can define a smooth vector field
\[ n \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} : U \longrightarrow TM \] by
\[ \left( n \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \right) (p) = n \sum_{i=1}^n f^i(p) \frac{\partial}{\partial x^i} \bigg|_p, \]
We call \( (f^1, \ldots, f^n) \) the *components* of the vector field \( \sum_{i=1}^n f^i \partial/\partial x^i \) with respect to the coordinates \((x^1, \ldots, x^n)\). Note that any vector field $X$ on $U$ is of the form
\[ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \]
where \( f^i = \dot{x}^i \circ X : U \rightarrow \mathbb{R} \) and \( \dot{x}^1, \ldots, \dot{x}^n \) are the velocity coordinates on $\pi^{-1}(U) \subseteq TM$.

The notation
\[ X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \tag{1.16} \]
correctly suggests that a vector field on the domain $U$ of a smooth coordinate system \((U, \phi) = (U, (x^1, \ldots, x^n))\) can be regarded as a differential operator. Indeed, if $\mathcal{F}(U)$ denote the space of smooth real-valued functions on $U$, then a vector field $X$ on $U$ defines a first-order differential operator
\[ \dot{X} : \mathcal{F}(U) \rightarrow \mathcal{F}(U) \] by
\[ \dot{X}(f)(p) = X(p)(f). \]
This definition makes because \((f, U)\) can be regarded as an element of $\mathcal{F}(p)$ when $p \in U$. To see that \( \dot{X}(f) \) is smooth, we note that if $X$ is written in the form (1.16) and $g \in \mathcal{F}(U)$,
\[ \dot{X}(g) = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}(g) = \sum_{i=1}^n (\dot{x}^i \circ X)(D_i (g \circ \phi^{-1}) \circ \phi), \]
$D_i$ being the $i$-th partial derivative in Euclidean space, so $\dot{X}(g)$ is smooth. The operator $\dot{X}$ satisfies the axioms:

1. \( \dot{X}(cf + g) = c \dot{X}(f) + \dot{X}(g) \), for $c \in \mathbb{R}$ and $f, g \in \mathcal{F}(U)$,
2. $\hat{X}(fg) = f\hat{X}(g) + g\hat{X}(f)$, for $f, g \in \mathcal{F}(U)$.

Conversely, any operator $\hat{X} : \mathcal{F}(U) \to \mathcal{F}(U)$ satisfying these axioms defines a corresponding vector field

$$X = \sum_{i=1}^{n} \hat{X}(x^i) \frac{\partial}{\partial x^i}.$$ 

To simplify notation, we will usually drop the hat on the differential operator $\hat{X}$ which corresponds to the vector field $X$.

Similarly, we can define smooth one-forms $dx^i : U \to T^* M$ by

$$dx^i(p) = dx^i|_p,$$

for $1 \leq i \leq n$, and if $f_1, \ldots, f_n : U \to \mathbb{R}$ are smooth functions on $U$, we can define a smooth one-form

$$\sum_{i=1}^{n} f_i dx^i : U \to T^* M \quad \text{by} \quad \left(\sum_{i=1}^{n} f_i dx^i\right)(p) = \sum_{i=1}^{n} f_i(p) dx^i|_p.$$ 

We call $(f_1, \ldots, f_n)$ the components of the one-form $\sum_{i=1}^{n} f_i dx^i$ with respect to the coordinates $(x^1, \ldots, x^n)$. Any smooth one-form on $U$ is of the form

$$\omega = \sum_{i=1}^{n} f_i dx^i \quad \text{where} \quad f_i = p_i \circ X : U \to \mathbb{R},$$

and $(p_1, \ldots, p_n)$ are the momentum coordinates on $\pi^{-1}(U) \subseteq T^* M$.

Vector fields can be regarded as systems of ordinary differential equations. Indeed, suppose that $X : M \to TM$ is a smooth vector field on $M$. We say that a smooth curve $\gamma : (a, b) \to M$ is an integral curve for $X$ if

$$\gamma'(t) = X(\gamma(t)), \quad \text{for} \ t \in (a, b). \quad (1.17)$$

We can write this in terms of a local coordinate system $(U, \phi)$, where $\phi = (x^1, \ldots, x^n)$, in terms of which $X$ is expressed as

$$X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}.$$ 

Since

$$\gamma'(t) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(t) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)} \quad \text{and} \quad X(\gamma(t)) = \sum_{i=1}^{n} f^i(\gamma(t)) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)},$$

then (1.17) becomes

$$\frac{d(x^i \circ \gamma)}{dt}(t) = f^i(\gamma(t)), \quad 1 \leq i \leq n.$$
If we define a function \( \tilde{f}^i \) on \( \phi(U) \) by \( \tilde{f}^i = f^i \circ \phi^{-1} \), we can rewrite this as

\[
\frac{d(x^i \circ \gamma)}{dt}(t) = \tilde{f}^i(x^1 \circ \gamma(t), \ldots, x^n \circ \gamma(t)), \quad 1 \leq i \leq n,
\]

which is just a system of ordinary differential equations. The integral curves for \( X \) correspond to the solutions of this system.

**Smooth one-forms can be regarded as integrands for line integrals.** To define the line integral, we need the important fact that smooth one-forms pull back under smooth maps. If \( F : M \to N \) is a smooth map and \( \omega \) is a smooth one-form on \( N \), we can define a smooth one-form \( F^* \omega \) on \( M \) by

\[
(F^* \omega)(p) = F^*_p(\omega(F(p))), \quad \text{for } p \in M.
\]

Although vector fields can be pushed forward under smooth diffeomorphisms, unfortunately there is no mechanism for pushing a vector field forward under an arbitrary smooth map.

Suppose that \( \gamma : [a, b] \to M \) is a smooth parametrized curve. If \( \omega \) is a smooth one-form on \( M \), we can regard \( \gamma^* \omega \) as a smooth one-form on \( [a, b] \subseteq \mathbb{R} \) and write \( \gamma^* \omega = g(t)dt \), where \( g \) is a smooth real-valued function. (To be precise, we will see later that \( [a, b] \) is a smooth manifold with boundary with coordinate \( t \).) We are then able to define the **line integral** of \( \omega \) along \( \gamma \)

\[
\int_{\gamma} \omega \quad \text{to be} \quad \int_{[a, b]} \gamma^* \omega = \int_{a}^{b} g(t)dt,
\]

the last integral being the usual Riemann integral of \( g \) as studied in real analysis courses. (This idea will be generalized later in the course when we treat integral calculus on manifolds with boundary of arbitrary dimension.)

If \( f : M \to \mathbb{R} \) is a smooth function, the **differential** of \( f \) is the one-form \( df \) defined by

\[
(df)(p) = df|_p, \quad \text{for } p \in M.
\]

It is an immediate consequence of Proposition 1.5.4 that if \( F : M \to N \) is a smooth map and \( f : N \to \mathbb{R} \) is a smooth function, then \( F^*(df) = d(f \circ F) \). One can check that if \( \gamma : [a, b] \to M \) is a smooth parametrized curve,

\[
\int_{\gamma} df = \int_{[a, b]} \gamma^* df = \int_{[a, b]} d(f \circ \gamma) = \int_{a}^{b} \frac{d(f \circ \gamma)}{dt} dt = f(\gamma(b)) - f(\gamma(a)),
\]

where we have used the fundamental theorem of calculus in the last step.

**Exercise II.** (Due Wednesday, October 22.) As you know, in addition to the usual Euclidean coordinates \((x, y)\) on \( \mathbb{R}^2 \), it is often convenient to use polar coordinates \((r, \theta)\), which are defined on the subset

\[
U = \mathbb{R}^2 - \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}
\]
by
\[ x = r \cos \theta, \quad y = r \sin \theta. \]

a. What is the vector field
\[ X = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \]
which corresponds to the system of differential equations that is expressed in Euclidean coordinates by
\[ \begin{align*}
\frac{dx}{dt} &= x - y - x(x^2 + y^2), \\
\frac{dy}{dt} &= x + y - y(x^2 + y^2).
\end{align*} \]

b. Use the chain rule to express the vector field \( X \) in terms of \( \frac{\partial}{\partial r} \) and \( \frac{\partial}{\partial \theta} \), and write out the corresponding system of differential equations in polar coordinates.

c. Find an integral curve \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) for the vector field \( X \) which satisfies the initial conditions
\[ x \circ \gamma(0) = 1, \quad y \circ \gamma(0) = 0. \]
(Hint: Show that the curve \( r = 1 \) solves the differential equation for \( r \).)

d. Express the smooth one-form
\[ \omega = ydx - xdy \]
in terms of polar coordinates.

e. If \( \gamma : [0, 2\pi] \to \mathbb{R}^2 \) is defined by
\[ x \circ \gamma(t) = \cos t, \quad y \circ \gamma(t) = \sin t, \]
what is the line integral of \( \omega \) along \( \gamma \)?
Chapter 2

Submanifolds

2.1 Immersions and imbeddings

So far, we have emphasized the extrinsic point of view, but sometimes it is useful to regard smooth manifolds as subspaces of Euclidean space of some dimension. To do this idea justice, we need the theory of imbeddings of manifolds. We will see that the inverse function theorem from the calculus of several variables can often be used to show that certain subsets of Euclidean space are imbedded smooth submanifolds (and therefore smooth manifolds) without going through the laborious task of constructing an atlas of smooth charts. Our next goal is to describe how this goes, deferring for the moment the proof of the inverse function theorem itself.

Consider first how the linearization of a smooth map simplifies when the map has open subsets of Euclidean spaces as domain and range. Suppose that $(x^1, \ldots, x^n)$ are the standard Euclidean coordinates on $\mathbb{R}^n$ and $(y^1, \ldots, y^m)$ are the Euclidean coordinates on $\mathbb{R}^m$. If $F : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map, then we recall from §1.5 that its “linearization” $(F_*)_p : T_p \mathbb{R}^n \to T_{f(p)} \mathbb{R}^m$ is defined by

$$(F_*)_p \left( \frac{\partial}{\partial x^i} \bigg|_p \right) = \sum_{j=1}^m \frac{\partial (y^j \circ F)}{\partial x^i} (p) \frac{\partial}{\partial y^j} \bigg|_p.$$ 

Thus if we identify $T_p \mathbb{R}^n$ with $\mathbb{R}^n$ and $T_{F(p)} \mathbb{R}^m$ with $\mathbb{R}^m$, then in terms of the standard bases, $(F_*)_p$ is just the matrix multiplication represented by the Jacobian matrix

$$\left( \frac{\partial (y^j \circ F)}{\partial x^i} \right) (p).$$

We can use maps between Euclidean spaces to illustrate the types of smooth maps described in the following definition:

**Definition.** If $M$ and $N$ are smooth manifolds, a smooth map $F : M \to N$ is

1. an **immersion** if $(F_*)_p$ is injective for each $p \in M$,
2. a submersion if \((F_*)_p\) is surjective for each \(p \in M\),

3. an imbedding if it is a one-to-one immersion that maps \(M\) homeomorphically onto \(F(M)\) when \(F(M)\) is given the topology it inherits as a subspace of \(N\).

Finally, we say that a point \(q \in N\) is a regular value for \(F : M \to N\) if \((F_*)_p\) is surjective for each \(p \in F^{-1}(q)\).

**Example 1.** The map \(\gamma : \mathbb{R} \to \mathbb{R}^2\) defined by \(\gamma(t) = (t^2, t^3)\) is not an immersion, because 
\[
\gamma'(0) \left( \frac{d}{dt} \bigg|_0 \right) = 0 |_{\gamma(0)} \in T_{(0,0)}\mathbb{R}^2,
\]
and the vanishing derivative is accompanied by a singularity in the image. We say that this cubic curve has a cusp at the origin.

**Example 2.** The map \(\gamma : \mathbb{R} \to \mathbb{R}^2\) defined by \(\gamma(t) = (\cos t, \sin t)\) is an immersion but not one-to-one.

**Example 3.** The map \(\gamma : (-\pi, \pi) \to \mathbb{R}^2\) defined by \(\gamma(t) = (\sin t, (1/2) \sin 2t)\) is a one-to-one immersion but not an imbedding, because any neighborhood of 0 in the subspace topology must contain \((-\pi, -\pi + \varepsilon) \cup (\pi - \varepsilon, \pi)\) for some \(\varepsilon > 0\).

**Proposition 2.1.1.** If \(M\) is compact and \(F : M \to N\) is a one-to-one immersion, then \(F\) is an imbedding.

Sketch of proof: If \(C\) is a closed subset of \(M\), then \(C\) is compact because closed subsets of compact spaces are always compact. The image of a compact set under a continuous map is always compact, so \(F(C)\) is compact. But a compact subset of a Hausdorff space is always closed, so \(F(C)\) is closed. Thus \(F\) is a closed continuous bijection, and therefore a homeomorphism. (A more complete argument is given in [15]; see the proof of Theorem 26.3.)

**Definition.** Let \(N\) be an \(n\)-dimensional smooth manifold, and \(m \leq n\). A subset \(M \subseteq N\) is an \(m\)-dimensional imbedded submanifold of \(N\) if it satisfies the following condition: Given \(p \in M\), there is a smooth chart \((U, \phi)\) on \(N\) with \(p \in U\) such that
\[
\phi(U \cap M) = \phi(U) \cap (\mathbb{R}^m \times \{0\}). \tag{2.1}
\]
Here we regard \(\mathbb{R}^m \times \{0\}\) as the linear subspace of \(\mathbb{R}^n\) defined by the equations
\[
x^{m+1} = \cdots = x^n = 0.
\]
An \(m\)-dimensional imbedded submanifold of \(N\) into an \(m\)-dimensional smooth manifold in its own right. We can see this as follows: Note first that \(M\) inherits a Hausdorff topology with countable base from \(N\), so we only need to construct a smooth atlas. To construct the atlas, we let
\[
\pi : \mathbb{R}^n \to \mathbb{R}^m \quad \text{be the projection} \quad \pi(x^1, \ldots, x^n) = (x^1, \ldots, x^m)
\]

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onto the first $m$ coordinates. We can then take

$$\mathcal{A} = \{(U \cap M, \pi \circ \phi) : (U, \phi) \text{ is a smooth chart on } N \text{ satisfying (2.1)} \}$$

as a smooth atlas on $M$. Moreover, when $M$ is given this smooth structure, the inclusion map $\imath : M \to N$ is easily checked to be an imbedding.

**Important Special Case:** If $N = \mathbb{R}^3$, a two-dimensional imbedded submanifold is just would commonly be called an imbedded surface in $\mathbb{R}^3$. These objects, together with the curvature and so forth inherited from the imbedding in $\mathbb{R}^3$ are studied in undergraduate courses on the differential geometry of curves and surfaces.

We can also go the other direction, from an imbedding to a submanifold:

**Theorem 2.1.2.** If $F : M \to N$ is an imbedding, then $F(M)$ is an imbedded submanifold of $N$ and $F$ maps $M$ diffeomorphically onto $F(M)$.

We will see that this is a consequence of the inverse function theorem. In addition, we will prove the following consequence of the inverse function theorem:

**Theorem 2.1.3.** If $N$ and $Q$ are smooth manifolds, $F : N \to Q$ is a smooth map and $q \in Q$ is a regular value for $F$, then $M = F^{-1}(q)$ is an imbedded submanifold of $N$ which has dimension $\dim N - \dim Q$.

Note that this last theorem makes it possible for us to show that certain subsets of Euclidean space are smooth submanifolds without constructing the explicit charts.

For example, suppose that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function. Then we can identify

$$(f_*)_p : T_p\mathbb{R}^{n+1} \to T_{f(p)}\mathbb{R}$$

with the differential $df|_p$, and $(f_*)_p$ will be surjective if and only if $df|_p \neq 0$. Thus $q$ is a regular value for $f$ if and only if the linear map

$$df|_p : T_pM \to \mathbb{R}$$

is nonzero, whenever $f(p) = q$.

We can specialize this to the case where

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

is defined by

$$f(x^1, x^2, \ldots, x^{n+1}) = (x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2.$$ 

We calculate the differential of $f$, with the result

$$df = 2x^1dx^1 + 2x^2dx^2 + \cdots + 2x^{n+1}dx^{n+1},$$

from which we see that 1 is a regular value. Thus we recover from Theorem 2.1.3 the fact that $S^n = f^{-1}(1)$ is an $n$-dimensional smooth submanifold of $\mathbb{R}^{n+1}$. 

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Thus we can use Theorem 2.1.3 to show that $S^n$ is a smooth manifold without explicitly constructing the smooth charts!

We can also use this theorem to show that the group of rotations in Euclidean space is a smooth manifold. To do this, we first note that

$$\text{Mat}(n, \mathbb{R}) = \{ n \times n \text{ matrices with real entries } \}$$

is a vector space of dimension $n^2$, and hence can be regarded as a smooth manifold of dimension $n^2$ with an atlas consisting of one chart. Indeed, we can take the coordinates on $\text{Mat}(n, \mathbb{R})$ to be the functions

$$x^i_j : \text{Mat}(n, \mathbb{R}) \rightarrow \mathbb{R} \text{ defined by } x^i_j \left( \begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{array} \right) = a^i_j.$$  

Similarly, the linear subspace of symmetric matrices

$$\text{Sym}(n, \mathbb{R}) = \{ A \in \text{Mat}(n, \mathbb{R}) : A = A^T \}$$

is a real vector space, and hence a smooth manifold, of dimension $(1/2)n(n+1)$. We define a map

$$F : \text{Mat}(n, \mathbb{R}) \rightarrow \text{Sym}(n, \mathbb{R}) \text{ by } F(A) = A^T A,$$  

which is clearly smooth, and let

$$O(n) = \{ A \in \text{Mat}(n, \mathbb{R}) : A^T A = I \} = F^{-1}(I).$$

It is readily checked that $O(n)$ is a group under matrix multiplication, and we call it the orthogonal group. Finally,

$$SO(n) = \{ A \in O(n) : \det A \neq 0 \}$$

is also a group under matrix multiplication, called the special orthogonal group or the rotation group for $\mathbb{R}^n$.

In algebra courses, one shows that $A \in O(n) \Rightarrow \det A = \pm 1$.

Since the determinant is continuous, it follows that $SO(n) = \det^{-1}(1)$ is both open and closed as a subset of $O(n)$. Canonical form theorems show that elements of $SO(2)$ are of the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

which represents rotation through an angle $\theta$, while elements of $SO(3)$ are rotations through some angle about some axis. If $A \in O(2m)$, there is always an
orthonormal basis for $\mathbb{R}^n$ with respect to which $A$ has a block diagonal form,

$$
\begin{pmatrix}
cos \theta_1 & \sin \theta_1 & & \\
-\sin \theta_1 & cos \theta_1 & & \\
& & \ddots & \\
& & & cos \theta_m & \sin \theta_m \\
& & & -\sin \theta_m & cos \theta_m
\end{pmatrix},
$$

while if $A \in SO(2m+1)$ an orthonormal basis can be chosen which represents $A$ as

$$
\begin{pmatrix}
cos \theta_1 & \sin \theta_1 & & \\
-\sin \theta_1 & cos \theta_1 & & \\
& & \ddots & \\
& & & cos \theta_m & \sin \theta_m \\
& & & -\sin \theta_m & cos \theta_m
\end{pmatrix}.
$$

Using these canonical forms you can show that $SO(n)$ is path connected for all $n \geq 2$; indeed, you can multiply each $\theta_i$ by $t$ to get a path from an arbitrary element to the identity. We can therefore say that $SO(n)$ is the connected component of $O(n)$ which contains the identity element.

**Lemma 2.1.4.** The identity matrix $I$ is a regular value for $F$ defined by (2.2), and hence $O(n)$ and $SO(n)$ are submanifolds of $\text{Mat}(n, \mathbb{R})$ of dimension $(1/2)n(n-1)$.

To prove this, we note that any $A \in O(n)$ is invertible, so whenever $A \in O(n)$, the linear map

$$
L_A : \text{Mat}(n, \mathbb{R}) \longrightarrow \text{Mat}(n, \mathbb{R}) \text{ defined by } L_A(B) = AB
$$

is a diffeomorphism. Moreover, when $A \in O(n)$,

$$
F(L_AB) = (AB)^T(AB) = B^TA^TAB = B^TB = F(B).
$$

So $A \in O(n) \Rightarrow F \circ L_A = F$. Thus it follows from the chain rule that

$$
(F_*)_I = (F_*)_A \circ ((L_A)_*)_I, \quad \text{for } A \in O(n),
$$

and hence

$$
(F_*)_A \text{ is surjective for } A \in O(n) \iff (F_*)_I \text{ is surjective.}
$$

Thus to prove that $I$ is a regular value, we need only check that $(F_*)_I$ is surjective.
There are at least two ways of doing this. One way is to show that the dual map

\[ F^* : T^*_I \text{Sym}(n, \mathbb{R}) \rightarrow T^*_I \text{Mat}(n, \mathbb{R}) \]

is injective using the local coordinates \( x^i_j : \text{Mat}(n, \mathbb{R}) \rightarrow \mathbb{R} \) defined before. In terms of these coordinates,

\[
x^i_j \circ F = \sum_{k=1}^{n} x^k_i x^k_j \quad \Rightarrow \quad d(x^i_j \circ F) = \sum_{k=1}^{n} x^k_i dx^k_j + \sum_{k=1}^{n} dx^k_i x^k_j
\]

\[
\Rightarrow \quad F^*_I(dx^i_j|_I|) = dx^i_j|_I + dx^i_i|_I,
\]

and hence

\[
F^*_I(dx^i_j|_I + dx^i_i|_I) = 2(dx^i_j|_I + dx^i_i|_I).
\]

It is easily verified that

\[
\{dx^j_i|_I + dx^i_i|_I : 1 \leq i \leq j \leq n\}
\]

is a basis for \( T^*_I \text{Sym}(n, \mathbb{R}) \). Since \( F^*_I \) takes a basis for \( T^*_I \text{Sym}(n, \mathbb{R}) \) to a linearly independent set of vectors in \( T^*_I \text{Mat}(n, \mathbb{R}) \), we see that \( F^*_I \) is indeed injective, so its dual \((F^*)_I\) is surjective, and our claim is established.

There is a second method, even simpler, for showing that \( I \) is a regular value for \( F \), and this will be explained at the beginning of § 2.3.

### 2.2 Lie groups

The theory of smooth manifolds makes accessible an important part of group theory. In fact, the most important symmetry groups that arise in physics (for example, the rotation group, the Lorentz group, the unitary group) are all examples of Lie groups. Lie groups will be studied in more detail later in the course, but it is helpful to have the definition in mind from the very start.

**Definition.** A Lie group is a group \( G \) together with a smooth manifold structure such that the maps

\[
\mu : G \times G \rightarrow G, \quad \text{defined by} \quad \mu(\sigma, \tau) = \sigma \tau
\]

and

\[
\nu : G \rightarrow G \quad \text{defined by} \quad \nu(\sigma) = \sigma^{-1}
\]

are smooth maps.

Of course, the simplest Lie group is the group \( \mathbb{R}^n \) in which the group operation is vector addition. Another simple example is

\[
S^1 = \{ z \in \mathbb{C} : |z| = 1 \}
\]

under multiplication. Other examples can be constructed by taking products: if \( G \) and \( H \) are Lie groups, so is their direct product \( G \times H \).
However, the idea of Lie group is most clearly exemplified by the so-called general linear group and its subgroups. The general linear group is

\[ GL(n, \mathbb{R}) = \{ A \in \text{Mat}(n, \mathbb{R}) : \det A \neq 0 \} , \]

the set of all \( n \times n \) real invertible matrices, with matrix multiplication as the group operation and the identity matrix \( I \) as the multiplicative identity. This is simply an open subset of the vector space \( \text{Mat}(n, \mathbb{R}) \), and as smooth coordinates we can take the standard coordinates

\[
x^i_j : GL(n, \mathbb{R}) \to \mathbb{R} \quad \text{defined by} \quad x^i_j \begin{pmatrix} a^1_1 & \cdots & a^1_n \\ \vdots & \ddots & \vdots \\ a^n_1 & \cdots & a^n_n \end{pmatrix} = a^i_j .
\]

Since

\[
x^i_j(AB) = \sum_{k=1}^{n} x^i_k(A)x^k_j(B),
\]

we see that matrix multiplication

\[ \mu : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R}) \]

is indeed smooth. On the other hand, the fact that

\[
x^i_j(A^{-1}) = \frac{1}{\det A} ( \text{the } (j, i)\text{-cofactor of } A ),
\]

we see that the inverse map \( \nu : GL(n, \mathbb{R}) \to GL(n, \mathbb{R}) \) is also smooth.

Most of the important Lie groups that arise in geometry are subgroups of the general linear group \( GL(n, \mathbb{R}) \). An example is the space of orthogonal matrices,

\[ O(n) = \{ A \in GL(n, \mathbb{R}) : A^T A = I \} . \]

As we saw in the previous section, \( O(n) \) is a subgroup of \( GL(n, \mathbb{R}) \) and also a submanifold of \( GL(n, \mathbb{R}) \); hence it is immediate that the multiplication and inverse maps

\[ \mu : O(n) \times O(n) \to GL(n, \mathbb{R}) \quad \text{and} \quad \nu : O(n) \to GL(n, \mathbb{R}) \]

are smooth. We need to check that these maps are smooth when the ranges of these maps are replaced by \( O(n) \), but this is an immediate consequence of the following lemma:

**Lemma 2.2.1.** Suppose that \( S \) is an imbedded submanifold of a smooth manifold \( N \) and that \( F : M \to N \) is a smooth map such that \( F(M) \subseteq S \). Then \( F : M \to S \) is also smooth.

The proof is relatively straightforward. To check smoothness at a point \( p \in M \), use charts \( (U, \phi) \) on \( M \) and \( (V, \psi) \) on \( N \) such that

\[
p \in U, \quad F(p) \in V, \quad \psi(S \cap V) = \psi(V) \cap (\mathbb{R}^k \times \{0\}),
\]

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where $k$ is the dimension of $S$. If $F: M \to N$ is smooth, then $\psi \circ F \circ \phi^{-1}$ is smooth, and hence $\pi \circ \psi \circ F \circ \phi^{-1}$ is smooth, where $\pi$ is the projection to $\mathbb{R}^k$. But $(S \cap V, \pi \circ \psi)$ is by definition one of the smooth charts defining the smooth structure on $S$. Hence $F: M \to S$ is indeed smooth. QED

Thus $O(n)$ is as expected a Lie group. We say that $O(n)$ is a Lie subgroup of $GL(n, \mathbb{R})$.

Note that if $A \in GL(n, \mathbb{R})$, then $Q = A^T A$ is a positive-definite symmetric matrix, and it is possible to define a square root $P = Q^{1/2}$ of this matrix. (Choose an invertible matrix $B$ such that $BQB^{-1}$ is diagonal, define the square root $BPB^{-1}$ by taking the square root of the diagonal entries, and finally conjugate this square root to obtain $P$.) One can check quite quickly that $U = AP^{-1}$ is an orthogonal matrix. Thus if we let $\text{Sym}^+(n, \mathbb{R})$ denote the space of positive-definite symmetric $n \times n$ matrices, we find that

$$A \in GL(n, \mathbb{R}) \quad \Rightarrow \quad A = UP, \quad \text{where} \quad U \in O(n), \quad P \in \text{Sym}^+(n, \mathbb{R}),$$

where $\text{Sym}^+(n, \mathbb{R})$ denotes the subset of positive-definite symmetric matrices within $\text{Sym}(n, \mathbb{R})$. This is known in linear algebra books as the polar decomposition of the nonsingular matrix $A$, and it is not too difficult to prove that it is unique. Thus as topological spaces,

$$GL(n, \mathbb{R}) \quad \text{is homeomorphic to} \quad O(n) \times \text{Sym}^+(n, \mathbb{R}),$$

where $\text{Sym}^+(n, \mathbb{R})$ is a convex open subset of $\mathbb{R}^N$, $N = (1/2)n(n + 1)$.

The previous ideas can be fruitfully generalized to complex and quaternionic matrices. Thus we can let $\text{Mat}(n, \mathbb{C})$ denote the space of $n \times n$ complex matrices, and let

$$GL(n, \mathbb{C}) = \{ A \in \text{Mat}(n, \mathbb{C}) : \det A \neq 0 \}.$$

In this case, we can define complex coordinates

$$z_j^i : GL(n, \mathbb{C}) \to \mathbb{C} \quad \text{defined by} \quad z_j^i \left( \begin{array}{cccc} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \cdots & a_n^n \end{array} \right) = a_j^i,$$

where now each $a_j^i$ is a complex number. This time the smooth coordinates on $GL(n, \mathbb{C})$ will be $x_j^i$ and $y_j^i$, where

$$z_j^i = x_j^i + \sqrt{-1} y_j^i.$$

Nevertheless, once again, the formulae for matrix multiplication and matrix inversion show that $GL(n, \mathbb{C})$ is a Lie group.

One of the most important subgroups of $GL(n, \mathbb{C})$ is the unitary group

$$U(n) = \{ A \in GL(n, \mathbb{C}) : A^T A = I \}.$$

Here $\bar{A}$ is the matrix obtained from $A$ by simply taking the complex conjugate of each of its entries. It is easily checked that $U(n)$ is indeed a subgroup of
GL(n, C); is it also an imbedded submanifold? To answer this question, we consider the real vector space

\[ \text{Herm}(n, \mathbb{C}) = \{ A \in \text{Mat}(n, \mathbb{C}) : A^T = \bar{A} \} \]

of Hermitian matrices and the smooth map

\[ F : \text{GL}(n, \mathbb{C}) \rightarrow \text{Herm}(n, \mathbb{C}) \quad \text{defined by} \quad F(A) = \bar{A}^T A. \]  

(2.3)

**Exercise III.** (Due Wednesday, October 29.) Show that \( I \) is a regular value for the map \( F \) defined by (2.3), and use this to conclude that the unitary group \( U(n) \) is indeed a Lie group.

**Exercise IIIA.** (Do not hand in.) Let \( SU(n) = \{ A \in U(n) : \det A = 1 \} \).

Show that \( SU(n) \) is a Lie group. (Hint: Show that 1 is a regular value for the determinant map \( \det : U(n) \rightarrow S^1 \).)

We can also develop a general linear group based upon the quaternions. The space \( \mathbb{H} \) of quaternions can be regarded as the space of complex \( 2 \times 2 \) matrices of the form

\[ Q = \begin{pmatrix} t + iz & x + iy \\ -x + iy & t - iz \end{pmatrix}, \]  

(2.4)

where \((t, x, y, z) \in \mathbb{R}^4 \) and \( i = \sqrt{-1} \). As a real vector space, \( \mathbb{H} \) is generated by the four matrices

\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \]

the matrix product restricting to the cross product on the subspace spanned by \( i, j \) and \( k \). Thus, for example, \( ij = k \) in agreement with the cross product. The **conjugate** of a quaternion \( Q \) defined by (3.15) is

\[ \bar{Q} = \begin{pmatrix} t - iz & -x - iy \\ x - iy & t + iz \end{pmatrix}, \]

and

\[ \bar{Q}^T Q = (t^2 + x^2 + y^2 + z^2)I = \langle Q, Q \rangle I, \]

where \( \langle , \rangle \) denotes the Euclidean dot product on \( \mathbb{H} \). Under the operations matrix addition and multiplication, the space \( \mathbb{H} \) of quaternions forms a skew-field, that is all of the field axioms are satisfied except for commutativity of multiplication.

We can let \( \text{Mat}(n, \mathbb{H}) \) denote the space of \( n \times n \) matrices with quaternion entries, and define a real linear conjugation

\[ C : \text{Mat}(n, \mathbb{H}) \rightarrow \text{Mat}(n, \mathbb{H}) \quad \text{by} \quad C(A) = \bar{A}, \]
where $\bar{A}$ is obtained by conjugating each each quaternion entry of $A$. The quaternion general linear group is then

$$GL(n, \mathbb{H}) = \{ A \in \text{Mat}(n, \mathbb{H}) : L_A \text{ is a diffeomorphism } \},$$

where

$$L_A : \text{Mat}(n, \mathbb{H}) \rightarrow \text{Mat}(n, \mathbb{H}) \text{ by } L_A(B) = AB.$$ 

The representation (3.15) shows how this can be regarded as a subgroup of $GL(2n, \mathbb{C})$. Finally, we can define the compact symplectic group

$$Sp(n) = \{ A \in GL(n, \mathbb{H}) : \bar{A}^T A = C(A)^T A = I \}.$$

If we let

$$\text{Herm}(n, \mathbb{H}) = \{ A \in \text{Mat}(n, \mathbb{H}) : A^T = \bar{A} \},$$

then $Sp(n) = F^{-1}(I)$, where

$$F : GL(n, \mathbb{H}) \rightarrow \text{Herm}(n, \mathbb{H}) \text{ defined by } F(A) = \bar{A}^T A. \quad (2.5)$$

**Exercise III B.** (Do not hand in.) Show that $I$ is a regular value for the map $F$ defined by (2.5), and use this to conclude that the compact symplectic group $Sp(n)$ is indeed a Lie group.

**Exercise III C.** (Do not hand in.) Suppose that $J$ is the $4 \times 4$ matrix

$$J = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

and let

$$O(1, 3) = \{ A \in GL(4, \mathbb{R}) : A^T JA = J \}.$$

Show that $J$ is a regular value of the map

$$F : \text{Mat}(n, \mathbb{R}) \rightarrow \text{Sym}(n, \mathbb{R}) \text{ by } F(A) = A^T JA,$$

and use this fact to show that $O(1, 3)$ is a Lie group. (This group is called the Lorentz group.)

### 2.3 The inverse function theorem

We digress briefly to discuss one of the key theorems of several variable calculus, the inverse function theorem. As background for doing this, we first describe the development of differential calculus of several variables in a coordinate-invariant notation, including a review of the notion of smooth map between Euclidean spaces.
Suppose that $U$ is an open subset of $\mathbb{R}^n$ and that $F : U \to \mathbb{R}^m$ is a continuous map. We say that $F$ is differentiable at $x_0 \in U$ if there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$F(x_0 + h) = F(x_0) + Th + o(h), \quad \text{for all } h \in \mathbb{R}^n \text{ such that } x_0 + h \in U,$$

where the “error term” $o(h)$ satisfies

$$\lim_{\|h\| \to 0} \frac{\|o(h)\|}{\|h\|} = 0,$$

with $\| \cdot \|$ denotes the Euclidean norm. (In other words, given $\varepsilon > 0$, when $\|h\|$ is sufficiently small, $\|o(h)\| \leq \varepsilon \|h\|$.)

We write $T = DF(x_0)$ and call $DF(x_0) \in L(\mathbb{R}^n, \mathbb{R}^m) = \{ \text{linear transformations from } \mathbb{R}^n \text{ to } \mathbb{R}^m \}$ the derivative of $F$ at $x_0$. Note that if $T \in L(\mathbb{R}^n, \mathbb{R}^m)$, the norm of $T$ is defined by the formula

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in \mathbb{R}^n - \{0\} \right\}.$$

Alternatively, $\|T\|$ is the smallest number such that

$$\|Tx\| \leq \|T\| \|x\|, \quad \text{for all } x \in \mathbb{R}^n.$$

(We can define a metric

$$d : L(\mathbb{R}^n, \mathbb{R}^m) \times L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}, \quad d(T_1, T_2) = \|T_1 - T_2\|,$$

and use this to define the topology on $L(\mathbb{R}^n, \mathbb{R}^m)$. This topology is the same as that obtained by regarding $L(\mathbb{R}^n, \mathbb{R}^m)$ as isomorphic to $\mathbb{R}^{nm}$ with the Euclidean dot product.)

We say that $F$ is $C^1$ on $U$ if $F$ is differentiable at every point $x_0 \in U$, and the map

$$DF : U \to L(\mathbb{R}^n, \mathbb{R}^m),$$

is continuous. For $k \geq 2$, we say that $F$ is $C^k$ if $F$ is $C^1$ and

$$DF : U \to L(\mathbb{R}^n, \mathbb{R}^m)$$

is $C^{k-1}$, and that $F$ is $C^\infty$ or smooth if it is $C^k$ for all $k$.

**Example.** Thinking of the derivative as a linear map (instead of focusing on the partial derivatives which are the components of the linear map) is a powerful and important idea. We can apply this idea to the map of (2.2),

$$F : \text{Mat}(n, \mathbb{R}) \to \text{Sym}(n, \mathbb{R}) \quad \text{by} \quad F(A) = A^T A.$$
Indeed, if we think of $B$ as a small matrix, then since
\[ F(A + B) = A^T A + (A^T B + B^T A) + B^T B \]
we see that
\[ DF(A)B = A^T B + B^T A, \quad \text{so} \quad DF(I)B = B + B^T, \]
and since any symmetric matrix is of the form $B + B^T$, we see that $DF(I)$ is surjective without using coordinates, providing an alternate approach to showing that $O(n)$ is a submanifold of $\text{Mat}(n, \mathbb{R})$. (Of course, we still need the first part of the earlier argument, which uses the diffeomorphism $L_A$ to show that $DF(I)$ surjective implies that $I$ is a regular value.)

Similarly, the derivatives of the maps
\[ F : \text{GL}(n, \mathbb{C}) \rightarrow \text{Herm}(n), \quad F(A) = \bar{A}^T A \]
and
\[ F : \text{GL}(n, \mathbb{H}) \rightarrow \text{Herm}(n, \mathbb{H}), \quad F(A) = \bar{A}^T A \]
at the identity $I$ can be shown to be given by the formula
\[ DF(I)B = \bar{B} + B^T. \]
Note that in the latter cases, $DF(I)$ is real linear but not complex or quaternionic linear. Since any Hermitian matrix is of the form $\bar{B} + B^T$, we see that $DF(I)$ surjective.

**Proposition 2.3.1. (The Chain Rule.)** If $U$ and $V$ are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, and
\[ F : U \rightarrow V \quad \text{and} \quad G : V \rightarrow \mathbb{R}^m \]
are smooth maps, then so is $G \circ F$ and
\[ D(G \circ F)(x_0) = DG(F(x_0))DF(x_0). \quad (2.6) \]

Sketch of proof: If $x_0 \in U$, then
\[ F(x_0 + h) = F(x_0) + DF(x_0)h + o(h), \]
\[ G(F(x_0) + k) = G(F(x_0)) + DG(F(x_0))k + o(k), \]
and hence
\[ GF(x_0 + h) = G(F(x_0) + DF(x_0)h + o(h)) \]
\[ = G(F(x_0)) + DG(F(x_0)) (DF(x_0)h + o(h)) + o(DF(x_0)h + o(h)), \]
Finally, one notes that
\[ DG(F(x_0))o(h) + o(DF(x_0)h + o(h)) = o(h), \]
which implies (2.6). QED

Using (2.6) and the Leibniz rule for differentiating a product, we can show that
\[ F \text{ and } G \text{ are } C^k \Rightarrow G \circ F \text{ is } C^k. \]

We can now state the inverse function theorem:

**Theorem 2.3.2. (Inverse Function Theorem.)** If \( U_1 \) and \( U_2 \) are open subsets of \( \mathbb{R}^n \) with \( x_0 \in U_1 \), and \( F : U_1 \to U_2 \) is a \( C^\infty \) map such that \( DF(x_0) \in L(\mathbb{R}^n, \mathbb{R}^n) \) is invertible, then there are open neighborhoods \( V_1 \) of \( x_0 \) and \( V_2 \) of \( F(x_0) \) such that
\[ F(V_1) = V_2, \text{ and a } C^\infty \text{ map } G : V_2 \to V_1 \text{ such that} \]
\[ F \circ G = id_{V_2} \quad \text{and} \quad G \circ F = id_{V_1}. \]

Moreover, \( DG(F(x)) = [DF(x)]^{-1} \), for \( x \in V_1 \).

The proof is based upon:

**Lemma 2.3.3. (Contraction Mapping Lemma.)** Let \((X,d)\) be a complete metric space and suppose that \( H : X \to X \) is a mapping such that
\[ d(H(x), H(y)) \leq \frac{1}{2} d(x, y), \quad \text{for } x, y \in X. \quad (2.7) \]

Then \( H \) has a unique fixed point.

**Remark.** Actually, the only case we need for the proof is the one in which \( X \) is a closed ball in \( \mathbb{R}^n \) and \( d(x, y) = \|x - y\| , \) for \( x, y \in \mathbb{R}^n \).

**Proof:** Note that (2.8) implies that \( H \) is continuous. Choose \( x_0 \in X \) and let \( x_n = H(x_{n-1}) \) for \( n \geq 1 \). Let \( D = d(x_0, x_1) \). Then it follows from (2.8) that
\[ d(x_{n+1}, x_n) \leq \left( \frac{1}{2} \right)^n D. \]

Moreover, if \( m > n \), then
\[ d(x_n, x_m) \leq \left[ \left( \frac{1}{2} \right)^n + \left( \frac{1}{2} \right)^{n+1} + \cdots + \left( \frac{1}{2} \right)^{m-1} \right] D \]
\[ \leq \left( \frac{1}{2} \right)^n 2D \leq \left( \frac{1}{2} \right)^{n-1} D. \]

Thus \((x_n)\) is a Cauchy sequence in \((X,d)\) and since \((X,d)\) is complete, the Cauchy sequence converges to a limit \( x_\infty \in X \). Moreover, by continuity of \( H \),
\[ d(x_\infty, H(x_\infty)) = \lim_{n \to \infty} d(x_n, H(x_n)) = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0, \]

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so $H(x_\infty) = x_\infty$, so $x_\infty$ is a fixed point for $H$.

If $x_\infty$ and $x'_\infty$ are two fixed points for $H$, then by (2.8),
\[
d(x_\infty, x'_\infty) = d(H(x_\infty), H(x'_\infty)) \leq \frac{1}{2} d(x_\infty, x'_\infty) \quad \Rightarrow \quad d(x_\infty, x'_\infty) = 0,
\]
so $x_\infty = x'_\infty$. QED

Sketch of proof of the Inverse Function Theorem: We can assume without loss of generality that $x_0 = 0 \in U_1$ and $F(0) = 0 \in U_2$. We can assume, moreover, that $DF(0)$ is the identity map by replacing $F$ by $DF(0)^{-1} \circ F$. For a given $y$ which is close to zero in $U_2$, we need to construct a contraction which has a fixed point $x$ such that $F(x) = y$. We choose $\delta > 0$ so small that the closed ball of radius $\delta$ about zero in $\mathbb{R}^n$ is contained in both $U_1$ and $U_2$.

Given $y \in U_2$ with $\|y\| < \delta/2$, we define maps $H : U_1 \to \mathbb{R}^n$ and $H_y : U_1 \to \mathbb{R}^n$ by $H(x) = x - F(x)$ and $H_y = H(x) + y$. Then
\[
H_y(x) = x \quad \Leftrightarrow \quad x - F(x) + y = x \quad \Leftrightarrow \quad F(x) = y,
\]
so that the preimages of $y$ are exactly the fixed points of $H_y$. Our goal is to show that the restriction of $H_y$ to a sufficiently small ball about zero is a contraction.

Note first that $DH_y(0) = DH(0) = I - DF(0) = 0$, and by continuity of $DH$ we can contract $\delta > 0$ so that
\[
\|x\| \leq \delta \quad \text{and} \quad \|y\| \leq \frac{\delta}{2} \quad \Rightarrow \quad x \in U_1 \quad \text{and} \quad \|dH_y(x)\| = \|dH(x)\| < \frac{1}{2}.
\]
If $\|x_1\|, \|x_2\| \leq \delta$, then it follows from the chain rule that
\[
\|H_y(x_1) - H_y(x_2)\| = \left\| \int_0^1 \frac{d}{dt}(H_y(tx_1 + (1-t)x_2))dt \right\|
\leq \left[ \int_0^1 \|DH_y(tx_1 + (1-t)x_2)(x_1 - x_2)dt \right] \|x_1 - x_2\| < \frac{1}{2}\|x_1 - x_2\|.
\]
In particular, if $\|x\| \leq \delta$,
\[
\|H_y(x) - H_y(0)\| = \|H_y(x) - y\| < \frac{1}{2}\|x - 0\| = \frac{\delta}{2}
\Rightarrow \quad \|H_y(x)\| < \|y\| + \frac{\delta}{2} \leq \delta,
\]
so $H_y$ takes the closed ball of radius $\delta$ into the open ball of radius $\delta$. It follows from (2.8) that $H_y$ is a contraction on the closed ball of radius $\delta$.

Thus by the Contraction Mapping Lemma, given $y$ with $\|y\| < \delta/2$, there is a unique fixed point $x$ of $H_y$ with $\|x\| < \delta$ and $\|x\| < \delta$; that is, there is a unique $x$ such that $\|x\| < \delta$ and $F(x) = y$. We can now let
\[
V_2 = \{ y \in \mathbb{R}^n : \|y\| < \delta/2 \} \quad \text{and} \quad V_1 = \{ x \in \mathbb{R}^n : \|x\| < \delta, F(x) \in V_2 \},
\]

and define

\[ G : V_2 \to V_1 \quad \text{by} \quad G(y) = x \in V_1 \iff F(x) = y. \]

Then \( G \) is a set-theoretic inverse to \( F : V_1 \to V_2 \).

To show that \( G \) is continuous, it suffices to show that

\[ \|x_1 - x_2\| \leq 2\|F(x_1) - F(x_2)\|. \]

But

\[ \|x_1 - x_2\| \leq \|(x_1 - F(x_1)) - (x_2 - F(x_2))\| + \|F(x_1) - F(x_2)\| \]
\[ \leq \|H(x_1) - H(x_2)\| + \|F(x_1) - F(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\| + \|F(x_1) - F(x_2)\|, \]

which clearly implies the desired result.

To see that that \( G \) is \( C^1 \), we note first that if \( x_0 \in V_1 \),

\[ F(x) - F(x_0) = DF(x_0)(x - x_0) + o(x - x_0). \]

If \( y_0 = F(x_0) \) and \( y = F(x) \), we can rewrite this equation as

\[ y - y_0 = DF(x_0)(G(y) - G(y_0)) + o(x - x_0), \]

or \( G(y) - G(y_0) = [DF(x_0)]^{-1}[(y - y_0) - o(x - x_0)] \).

The continuity argument shows that \([DF(x_0)]^{-1}(o(x - x_0))\) is \( o(y - y_0) \), so \( G \) is \( C^1 \) with derivative \( DG(y) = (DF)^{-1}(G(y)) \) or \( DG(F(x)) = (DF)^{-1}(x) \).

Finally, one uses “bootstrapping” to show that \( G \) is smooth:

\[ G \in C^1 \Rightarrow [DF \circ G]^{-1} \in C^1 \Rightarrow DG \in C^1 \Rightarrow G \in C^2 \Rightarrow \cdots. \]

By induction one establishes that \( G \) is \( C^k \) for all \( k \), thus finishing the proof of the Inverse Function Theorem.

The proof of the Inverse Function Theorem is important for several reasons. First the idea of using the Contraction Mapping Lemma is useful in other contexts, such as showing that ordinary differential equations in standard form have solutions. Second, the technique of bootstrapping used here, showing first that a solution is continuous, then \( C^1 \), then \( C^2 \), and so forth, is a fundamental technique used in the study of nonlinear partial differential equations.

Remark. The notion of derivative we have given extends to maps between infinite-dimensional Banach spaces, and the proof of the Inverse Function Theorem extends with no difference in the proof. This leads to the possibility of extending the theory of smooth manifolds to infinite dimensions, as developed by Eells, Smale and others, and described in an important book on infinite-dimensional manifolds by Lang [9]. This point of view has important applications within the theory of differential equations.
Corollary 2.3.4. Let $U$ be an open neighborhood of $0 \in \mathbb{R}^m$ and let $F: U \to \mathbb{R}^n$ be a smooth map such that $F(0) = 0$. If $DF(0)$ is injective, there is a diffeomorphism $G$ from one neighborhood of $0 \in \mathbb{R}^n$ onto another such that

$$(G \circ F)(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

To prove this, we let

$$F = \begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix},$$

where each $f^i$ is a real-valued function. After possibly reordering the coordinates, we can assume without loss of generality that

$$\begin{pmatrix} \left(\frac{\partial f^1}{\partial x^1}\right)(0) & \left(\frac{\partial f^1}{\partial x^m}\right)(0) \\ \vdots & \vdots \\ \left(\frac{\partial f^n}{\partial x^1}\right)(0) & \left(\frac{\partial f^n}{\partial x^m}\right)(0) \end{pmatrix}$$

is nonsingular, and define

$$\tilde{F}: U \times \mathbb{R}^{n-m} \to \mathbb{R}^n \text{ by } \tilde{F}(x^1, \ldots, x^n) = F(x^1, \ldots, x^m) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x^{m+1} \\ \vdots \\ x^n \end{pmatrix}.$$

It is then the case that

$$(D\tilde{F})(0) = \begin{pmatrix} \left(\frac{\partial f^1}{\partial x^1}\right)(0) & \left(\frac{\partial f^1}{\partial x^m}\right)(0) & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \left(\frac{\partial f^n}{\partial x^1}\right)(0) & \left(\frac{\partial f^n}{\partial x^m}\right)(0) & \ldots & 0 \\ \ast & \ast & \ldots & 1 \\ \vdots & \vdots & \ldots & \vdots \\ \ast & \ast & \ast & 0 \end{pmatrix}.$$ 

Hence $(D\tilde{F})(0)$ is invertible and it follows from the Inverse Function Theorem that $\tilde{F}$ possesses a local inverse $G$, and

$$(G \circ F)(x^1, \ldots, x^m) = (G \circ \tilde{F})(x^1, \ldots, x^m, 0, \ldots, 0) = (x^1, \ldots, x^m, 0, \ldots, 0) \quad QED$$

Corollary 2.3.5. Let $U$ be an open neighborhood of $0 \in \mathbb{R}^n$ and let $F: U \to \mathbb{R}^m$ be a smooth map such that $F(0) = 0$. If $DF(0)$ is surjective, there is a diffeomorphism $G$ from one neighborhood of $0 \in \mathbb{R}^n$ onto another such that

$$(F \circ G)(x^1, \ldots, x^n) = (x^1, \ldots, x^n).$$
To prove this, we once again let

$$F = \begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix},$$

where each $f^i$ is a real-valued function. After possibly reordering the coordinates, we can assume without loss of generality that

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1}(0) \cdot \frac{\partial f^1}{\partial x^m}(0) \\ \vdots \\ \frac{\partial f^m}{\partial x^1}(0) \cdot \frac{\partial f^m}{\partial x^m}(0) \end{pmatrix}$$

is nonsingular, and define

$$\tilde{F}: U \rightarrow \mathbb{R}^n \quad \text{by} \quad \tilde{F}(x^1, \ldots, x^n) = \begin{pmatrix} f^1(x^1, \ldots, x^n) \\ \vdots \\ f^m(x^1, \ldots, x^n) \\ x^{m+1} \\ \vdots \\ x^n \end{pmatrix},$$

so that if $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the projection on the first $m$ coordinates, then $\pi \circ \tilde{F} = F$. It is then the case that

$$(D\tilde{F})(0) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(0) \cdot \frac{\partial f^1}{\partial x^m}(0) & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^m}{\partial x^1}(0) \cdot \frac{\partial f^m}{\partial x^m}(0) & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

Hence $(D\tilde{F})(0)$ is invertible and it follows from the Inverse Function Theorem that $\tilde{F}$ possesses a local inverse $G$, and

$$(F \circ G)(x^1, \ldots, x^n) = (\pi \circ \tilde{F} \circ G)(x^1, \ldots, x^n) = \pi(x^1, \ldots, x^n) = (x^1, \ldots, x^m). \quad QED$$

We can in turn use these corollaries to prove Theorems 2.1.2 and 2.1.3 from §2.1.

**Proof of Theorem 2.1.2:** We need to show that if $F: M \rightarrow N$ is an imbedding, then $F(M)$ is an imbedded submanifold of $N$. Suppose that $p \in M$ and that $q = F(p) \in N$. We need to construct a smooth chart $(W, \sigma)$ on $N$ with $q \in W$ such that

$$\sigma(W \cap F(M) = \sigma(W) \cap (\mathbb{R}^m \times \{0\}), \quad (2.9)$$
where $m$ is the dimension of $M$. Let $(U, \phi)$ and $(V, \psi)$ be charts on $M$ and $N$ respectively with $p \in U$, $q \in V$, $\phi(p) = 0$, $\psi(q) = 0$. Since $F$ is an immersion

$$(D(\psi \circ F \circ \phi^{-1})(\phi(p)))$$

has rank $m$. Therefore, by Corollary 1.9.5, there is a diffeomorphism $G$ from some open neighborhood of zero in $\mathbb{R}^n$ onto another, where $n$ is the dimension of $N$, such that

$$G \circ \psi \circ F \circ \phi^{-1})(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

This composition is defined on $\phi(U')$ where $U'$ is a possibly smaller neighborhood of $p$, chosen so that there is an open subset $W$ of $N$ such that $W \cap F(M) = F(U')$. (This can be done because $F$ maps $M$ diffeomorphically to $F(M)$ with the subspace topology.) If $W$ is a sufficiently small open neighborhood of $q$,

$$(W, \sigma) = (W, (G \circ \psi)|W)$$

is a smooth chart satisfying (2.9). QED

Proof of Theorem 2.1.3: We need to show that if $N$ and $Q$ are smooth manifolds, $F : N \to Q$ is a smooth map and $q \in Q$ is a regular value for $F$, then $F^{-1}(q)$ is an imbedded submanifold of $N$. Let $(U, \phi)$ and $(V, \psi)$ be charts on $N$ and $Q$ respectively with $p \in U$, $q \in V$, $\phi(p) = 0$, $\psi(q) = 0$. Since $q$ is a regular value for $F$,

$$D(\psi \circ F \circ \phi^{-1})(\phi(p))$$

has rank $m$, where $m$ is the dimension of $Q$. Therefore, by Corollary 1.9.6, there is a diffeomorphism $G$ from some open neighborhood of zero in $\mathbb{R}^n$ onto another, where $n$ is the dimension of $N$, such that

$$(\psi \circ F \circ \phi^{-1})G(x^1, \ldots, x^n) = (x^1, \ldots, x^m).$$

We now let $H : \mathbb{R}^n \to \mathbb{R}^n$ be the map which interchanges the first $m$ and last $n - m$ coordinates,

$$H(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = (x^{m+1}, \ldots, x^n, x^1, \ldots, x^m).$$

If $W$ is a sufficiently small neighborhood of $p \in N$, then

$$(W, \sigma) = (W, (H \circ G^{-1} \circ \phi)|W)$$

is a smooth chart. Moreover, $G^{-1} \circ \phi(p)$ has its first $m$ coordinates zero if and only if $F(p) = q$, so $H \circ G^{-1} \circ \phi(p)$ has its last $m$ coordinates zero if and only if $F(p) = q$. Thus

$$\sigma(W \cap (F^{-1}(q))) = \sigma(W) \cap (\mathbb{R}^{n-m} \times \{0\}),$$

and $M = F^{-1}(q)$ is indeed a smooth submanifold of $N$ and it has dimension $n - m$. QED

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2.4 Whitney’s imbedding theorem

One of the first theorems in the theory of manifolds was Whitney’s 1936 theorem that a smooth manifold of dimension \( n \) can be imbedded in \( \mathbb{R}^{2n+1} \). Actually, in 1944, Whitney was able to improve the dimension a little, thereby showing that there is a smooth imbedding of any \( n \)-dimensional manifold in \( \mathbb{R}^{2n} \). Dimension-wise, this is best possible, because it is known that \( \mathbb{R}P^n \) cannot be imbedded in \( \mathbb{R}^{2n-1} \) when \( n \) is a power of two, a fact proven via “characteristic classes” of Stiefel and Whitney in Milnor and Stasheff [13]. These results of Whitney started the new subject now known as differential topology.

We will prove Whitney’s original imbedding theorem under the additional assumption that \( M \) is compact. The techniques used, partitions of unity and Sard’s Theorem, are useful for many other purposes.

**Partitions of unity.** If \( M \) is a smooth manifold and \( f : M \to \mathbb{R} \) is a continuous function on \( M \), the **support** of \( f \) is the set

\[
\text{supp}(f) = \text{closure of } \{ p \in M : f(p) \neq 0 \}.
\]

**Definition.** Let \( \mathcal{U} = \{ U_\alpha : \alpha \in A \} \) be an open cover of a smooth manifold \( M \). A **smooth partition of unity** subordinate to \( \mathcal{U} \) is a collection \( \{ \psi_\alpha : \alpha \in A \} \) such that

1. \( \psi_\alpha : M \to [0, 1] \cap \mathbb{R} \) is a smooth function,
2. \( \text{supp}(\psi_\alpha) \subseteq U_\alpha \),
3. if \( p \in M \) there is an open neighborhood \( V \) of \( p \) such that \( \text{supp}(\psi_\alpha) \cap V \) is nonempty for only finitely many \( \alpha \in A \),
4. \( \sum_{\alpha \in A} \psi_\alpha(p) = 1 \), for any \( p \in M \).

Note that the third condition ensures that the sum in the last condition is finite.

**Theorem 2.4.1.** Given any open cover \( \mathcal{U} \) of a smooth manifold \( M \), there is a smooth partition of unity subordinate to \( \mathcal{U} \).

We prove this here only in the case where \( M \) is compact; the general case is treated in Chapter 2 of Lee [10].

**Lemma 2.4.2.** If \( p \in M \) and \( V \) is an open neighborhood of \( p \), there is a smooth function \( f : M \to [0, 1] \subseteq \mathbb{R} \) such that

1. \( f \equiv 1 \) on a neighborhood of \( p \), and
2. \( \text{supp}(f) \subseteq V \).

Proof Step I: We first define a map \( g_1 : \mathbb{R} \to \mathbb{R} \) by

\[
g_1(s) = \begin{cases} 
  e^{-1/s}, & \text{for } s > 0, \\
  0, & \text{for } s \leq 0.
\end{cases}
\]
This map is clearly $C^\infty$ except possibly at $s = 0$. But

$$
\frac{d}{ds}(e^{-1/s}) = p_1\left(\frac{1}{s}\right)e^{-1/s}, \quad \text{and by induction} \quad \frac{d^k}{ds^k}(e^{-1/s}) = p_k\left(\frac{1}{s}\right)e^{-1/s},
$$

where each $p_k$ is a polynomial for each $k \in \mathbb{N}$. Thus

$$
\lim_{s \to 0} \frac{d^k}{ds^k}(e^{-1/s}) = \lim_{s \to 0} p_k\left(\frac{1}{s}\right)e^{-1/s} = \lim_{t \to \infty} \frac{p_k(t)}{e^t} = 0,
$$

so $g_1$ is $C^\infty$ and

$$
\frac{d^k g}{ds^k}(0) = 0, \quad \text{for all } k \in \mathbb{N}.
$$

Proof Step II: We next define a $C^\infty$ function $g_2 : \mathbb{R} \to \mathbb{R}$ by $g_2(s) = g_1(s)g_1(1-s)$ a smooth function $g_2$ such that

$$
\text{supp}(g_2) \subseteq [0, 1] \quad \text{and} \quad g_2(s) > 0 \quad \text{for } s \in (0, 1).
$$

We then go on to define a $C^\infty$ function $g_3 : \mathbb{R} \to \mathbb{R}$ by

$$
g_3(s) = \frac{\int_0^s g_2(u)du}{\int_0^1 g_2(u)du},
$$

a function which satisfies

$$
g_3(s) = \begin{cases} 
0, & \text{for } s \leq 0, \\
(0, 1), & \text{for } 0 < s < 1, \\
1, & \text{for } s \geq 1.
\end{cases}
$$

Finally, we define a $C^\infty$ function $g_4 : \mathbb{R} \to \mathbb{R}$ by

$$
g_4(s) = \begin{cases} 
g_3(2 - |s|), & \text{for } |s| \leq 2, \\
0, & \text{for } |s| \geq 2.
\end{cases}
$$

a function which satisfies

$$
g_4(s) = \begin{cases} 
1, & \text{for } |s| \leq 1, \\
\in (0, 1), & \text{for } 1 < |s| < 2, \\
0, & \text{for } |s| \geq 2.
\end{cases}
$$

Proof Step III: Finally, for each $\varepsilon > 0$, we define $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by setting $h_\varepsilon(x) = g_4(\|x\|/\varepsilon)$. Then

$$
h_\varepsilon(x) = \begin{cases} 
1, & \text{for } \|x\| \leq \varepsilon, \\
0, & \text{for } \|x\| \geq 2\varepsilon.
\end{cases}
$$
Let \((U, \phi)\) be a smooth chart on \(M\) with \(p \in U \subseteq V\) and \(\phi(p) = 0\) and choose \(\varepsilon > 0\) so that
\[
\{ x \in \mathbb{R}^n : ||x|| \leq 3\varepsilon \} \subseteq \phi(U).
\]
We then define \(f : M \to [0, 1]\) by
\[
f(q) = \begin{cases} h_\varepsilon(\phi(q)), & \text{for } q \in U, \\ 0, & \text{for } q \notin U. \end{cases}
\]
The \(f\) is identically one on a neighborhood of \(p\) and its support is contained within \(V\). QED

**Lemma 2.4.3.** If \(K\) is a compact subset of a smooth manifold \(M\) and \(V\) is an open subset of \(M\) containing \(K\), there is a smooth function \(f : M \to [0, 1]\) such that

1. \(f > 0\) on \(K\), and
2. \(\text{supp}(f) \subseteq V\).

Indeed, if \(p \in K\), it follows from the previous lemma that there is a smooth function \(f_p : M \to [0, 1]\) such that \(f_p \equiv 1\) on a neighborhood of \(p\) and \(\text{supp}(f_p) \subseteq V\). Let
\[
U_p = \{ q \in M : f_p(q) > 0 \}.
\]
Since \(K\) is compact there are finitely many points \(\{p_1, \ldots, p_k\}\) within \(K\) such that \(K \subseteq U_{p_1} \cup \cdots \cup U_{p_k}\). If we set
\[
f = \frac{1}{k}(f_{p_1} + \cdots + f_{p_k}),
\]
we find that \(f > 0\) on \(K\) and \(\text{supp}(f) \subseteq V\). QED

**Lemma 2.4.4.** Let \(\{U_\alpha : \alpha \in A\}\) be an open cover of a smooth compact manifold \(M\). Then there is an open cover \(\{V_\alpha : \alpha \in A\}\) of \(M\) (with the same index set \(A\)) such that \(\overline{V_\alpha} \subseteq U_\alpha\).

Proof: For each \(p \in M\) choose an open neighborhood \(U_p\) of \(p\) such that \(\overline{U_p} \subseteq U_\alpha\) for some \(\alpha \in A\). By compactness of \(M\) there exist finitely many points such that
\[
M \subseteq U_{p_1} \cup \cdots \cup U_{p_k}.
\]
Let \(V_\alpha = \bigcup\{U_{p_i} : U_{p_i} \subseteq U_\alpha\}\). Then \(V_\alpha\) is open, \(\overline{V_\alpha} \subseteq U_\alpha\) and \(\{V_\alpha : \alpha \in A\}\) is an open cover of \(M\). QED

Proof of Theorem 2.4.1 when \(M\) is compact: Given an open cover \(\mathcal{U} = \{U_\alpha : \alpha \in A\}\) of \(M\), we can use Lemma 1.10.4 to construct an open cover \(\{V_\alpha : \alpha \in A\}\) such that \(\overline{V_\alpha} \subseteq U_\alpha\) for each \(\alpha \in A\). Since \(M\) is compact, finitely many of the \(V_\alpha\)'s cover \(M\) and if the index set \(A\) is infinite, we can set
\[
V_\alpha = \emptyset, \quad \text{which forces us to set } \psi_\alpha = 0
\]

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for all but finitely many $\alpha \in A$. Since each $\bar{V}_\alpha$ is compact, it follows from Lemma 2.4.3 that there is a smooth function $f_\alpha : M \to [0,1]$ such that $f_\alpha > 0$ on $\bar{V}_\alpha$, and $\text{supp}(f) \subseteq U_\alpha$. Let
\[ f = \sum_{\alpha \in A} f_\alpha, \quad \psi_\alpha = \frac{f_\alpha}{f}, \]
the sum being finite. Then $\{\psi_\alpha : \alpha \in A\}$ is a smooth partition of unity subordinate to $U$. QED

**Easy Whitney Theorem.** Partitions of unity give an easy proof of the first step toward the proof of Whitney’s Theorem:

**Theorem 2.4.5.** If $M$ is a compact smooth manifold, there is an imbedding $F : M \to \mathbb{R}^N$ for some choice of $N$.

To prove this, we cover $M$ by finitely many charts
\[ \{(U_1, (x_1^1, \ldots x_1^n)), \ldots, (U_m, (x_m^1, \ldots x_m^n))\}, \]
and let $\{\psi_1, \ldots, \psi_m\}$ be a smooth partition of unity subordinate to this covering. We then define
\[ F : M \to \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} \]
by
\[ F = ((\psi_1, \psi_1 x_1^1, \ldots, \psi_1 x_1^n), \ldots, (\psi_m, \psi_m x_m^1, \ldots, \psi_m x_m^n)). \]

In this formula, the component
\[ (\psi_i, \psi_i x_i^1, \ldots, \psi_i x_i^n) \]
is extended to be zero outside $U_i$. It will suffice to show that

1. $F$ is one-to-one, and
2. $F$ is an immersion,

since a one-to-one immersion is an imbedding by Proposition 2.1.1.

For the first assertion, note that if $F(p) = F(q)$, then $\psi_i(p) = \psi_i(q)$, for $1 \leq i \leq m$. We can assume without loss of generality that $\psi_1(p) = \psi_1(q) > 0$, and then both $p$ and $q$ lie within $U_1$. But then
\[ (\psi_1(p)x_1^1(p), \ldots, \psi_1(p)x_1^n(p)) = (\psi_1(q)x_1^1(q), \ldots, \psi_1(q)x_1^n(q)) \]
\[ \Rightarrow (x_1^1(p), \ldots, x_1^n(p)) = (x_1^1(q), \ldots, x_1^n(q)) \Rightarrow p = q. \]

For the second assertion, note that if $p \in M$, then $\psi_i(p) > 0$ for some $i$, $1 \leq i \leq m$, and since $((\pi_i \circ F)_*)_p$ injective implies $(F_*)_p$ injective, it suffices to show that $\pi_i \circ F$ is an immersion on $\{p \in M : \psi_i(p) > 0\}$, where
\[ \pi_i : \mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \]

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is the projection on the $i$-th factor. We can assume without loss of generality that $i = 1$, and to simplify the notation, we write

$$
\pi_1 \circ F = (\psi_1, \psi_1 x_1^1, \ldots, \psi_1 x_1^n) = (\psi, \psi x^1, \ldots, \psi x^n),
$$

with $(x^1, \ldots, x^n)$ being the local coordinates on $U_1$. But then the rank of the differential of $\pi_1 \circ F$ is just the rank of the matrix

$$
\begin{pmatrix}
\frac{\partial \psi}{\partial x^1}, x^1 + \psi & \frac{\partial \psi}{\partial x^n}, x^1 \\
\vdots & \vdots \\
\frac{\partial \psi}{\partial x^n}, x^n + \psi
\end{pmatrix}.
$$

But by subtracting suitable multiples of the first row from the other rows, we see that this matrix is row equivalent to

$$
\begin{pmatrix}
\frac{\partial \psi}{\partial x^1} & \frac{\partial \psi}{\partial x^n} \\
\psi & 0 \\
0 & \psi
\end{pmatrix},
$$

which clearly has rank $n$ on $\{p \in M : \psi(p) > 0\}$. Thus $\pi_1 \circ F$ is an immersion on $\{p \in M : \psi(p) > 0\}$, so $F$ itself is an immersion. QED

**Sard’s Theorem.** Roughly speaking, Sard’s Theorem states that most values of a smooth function $F : M \to N$ are regular values.

To make the notion of “most values” precise, we can say that a subset $C \subseteq \mathbb{R}^n$ has measure zero if for any $\varepsilon > 0$ it is contained in a countable sequence of cubes of total $n$-dimensional volume less than $\varepsilon$. (One can show that this agrees with the notion of Lebesgue measure zero as studied in real analysis courses.) We say that a subset $C \subseteq M$ of a smooth manifold $M$ has measure zero if for every smooth chart $(U, \phi)$ on $M$, $\phi(U \cap C)$ has measure zero in $\mathbb{R}^n$.

**Theorem 2.4.6. (Sard’s Theorem)** If $F : M \to N$ is a $C^k$ map and $k > \dim M - \dim N$, then the set of $q \in N$ which are not regular values has measure zero.

Alternatively, we can say that almost all $q \in N$ are regular values.

Note that there are actually two cases to Sard’s theorem, both of which turn out to be useful. If $\dim M \geq \dim N$, then every preimage of a regular point in $N$ is a submanifold by one of the corollaries of the Inverse Function Theorem. If, on the other hand, $\dim M < \dim N$, then $(F_*)_p$ can never be surjective, so Sard’s Theorem says that almost all $q \in N$ are not in the image of $F$ in this case.

Sard’s Theorem is one of the theorems we will not prove in the course, since the techniques are measure-theoretic and not closely related to our main lines of argument. A short proof (in the $C^\infty$ case) can be found in §3 of Milnor [12].

**Theorem 2.4.7. (Compact Whitney Theorem)** If $M$ is a compact smooth manifold of dimension $n$, there is an imbedding $F : M \to \mathbb{R}^{2n+1}$.
In view of Theorem 2.4.5, it suffices to show that if \( F : M \to \mathbb{R}^N \) is an imbedding and \( N > 2n + 1 \), then there is also an imbedding \( F_0 : M \to \mathbb{R}^{N-1} \).

Let \( u \) be a unit-length vector in \( \mathbb{R}^N \) and let

\[
\pi_u : \mathbb{R}^N \to \mathbb{R}^{N-1}_u = (\text{orthogonal complement to } u)
\]

be the orthogonal projection. We claim that if \( N > 2n + 1 \), then for almost all \( u \in S^{N-1} \subseteq \mathbb{R}^N \),

\[
\pi_u \circ F : M \to \mathbb{R}^{N-1}_u
\]

is an imbedding. As before, it will suffice to show that

1. \( \pi_u \circ F \) is one-to-one, and
2. \( \pi_u \circ F \) is an immersion,

for some choice of \( u \).

To prove the first assertion, note that if \( \pi_u \circ F \) is not one-to-one, there exist points \( p \) and \( q \) in \( M \), \( p \neq q \), such that

\[
\frac{F(p) - F(q)}{|F(p) - F(q)|} = u.
\]

In other words, \( u \) is in the image of the map

\[
G : M \times M \setminus \Delta \to S^{N-1}, \quad G(p, q) = \frac{F(p) - F(q)}{|F(p) - F(q)|},
\]

where \( \Delta \) is the diagonal in \( M \times M \),

\[
\Delta = \{(p, q) \in M \times M : p = q\}.
\]

But \( N > 2n + 1 \Rightarrow N - 1 > 2n \), so it follows from Sard’s Theorem that almost all \( u \in S^{N-1} \) are not in the image of \( G \), so for almost all \( u \in S^{N-1} \), \( \pi_u \circ F \) is one-to-one.

To prove the second assertion, note that \( F : M \to \mathbb{R}^N \) induces a map \( F_* : TM \to \mathbb{R}^N \) by

\[
F_*(v) = (F_p)_*(v) \in T_{F(p)} \mathbb{R}^N \cong \mathbb{R}^N, \quad \text{for } v \in T_p M.
\]

If \( \pi_u \circ F \) is not an imbedding, there exists a nonzero vector \( v \in TM \) such that

\[
\pi_u \circ F_*(v) = 0 \quad \text{or} \quad F_*(v) \text{ is parallel to } u.
\]

In other words, \( [u] \in \mathbb{R}P^{N-1} \) lies in the image of

\[
H : TM \setminus \{ \text{zero vectors} \} \to \mathbb{R}P^{N-1}, \quad H(v) = [F_*(v)].
\]

But \( N > 2n + 1 \Rightarrow N - 1 > 2n \), so it follows from Sard’s Theorem that almost all \( u \in \mathbb{R}P^{N-1} \) are not in the image of \( H \), so for almost all \( u \in S^{N-1} \), \( \pi_u \circ F \) is an immersion. QED
Remarks. The same argument can be used to prove that if $M$ is an $n$-dimensional compact smooth manifold, there is an immersion $F : M \rightarrow \mathbb{R}^{2n}$. In this case, we only need to establish the second assertion in the argument, which can be improved dimension-wise by one by considering the so-called unit tangent bundle

$$T^1M = \{ v \in TM : \|F_*(v)\| = 1 \}$$

To see that this is a submanifold of $TM$, simply note that the function

$$\| \cdot \| : TM - \{ \text{ zero vectors } \} \rightarrow (0, \infty)$$

is a submersion, so that it follows from Theorem 1.7.3 that the unit tangent bundle is a submanifold of dimension $2n - 1$. We can thus define a map

$$H : T^1M \rightarrow \mathbb{R}P^{N-1}, \quad H(v) = [F_*(v)].$$

Since we reduced the dimension of the domain by one, $N > 2n \Rightarrow N - 1 > 2n - 1$, so it follows from Sard’s Theorem that almost all $[u] \in \mathbb{R}P^{N-1}$ are not in the image of $H$. This allows us to conclude that for almost all $u \in S^{N-1}$, the projection

$$\pi_u \circ F : M \rightarrow \mathbb{R}u^{N-1}$$

is an immersion, so long as $N > 2n$.

These arguments can be modified to show that if $M$ is an $n$-dimensional compact smooth manifold, an arbitrary smooth map $F : M \rightarrow \mathbb{R}^N$ can be perturbed to an imbedding if $N \geq 2n + 1$, or an immersion if $N \geq 2n$, which are dimension-wise the best possible results.

Exercise IV. (Due Friday, November 10.) Show that the smooth map $F : \mathbb{R}P^n \rightarrow \mathbb{R}^{(n+1)^2}$ defined by

$$F([x^1, x^2, \ldots, x^{n+1}]) = \frac{1}{(x^1)^2 + \ldots + (x^{n+1})^2} \begin{pmatrix}
(x^1)^2 & x^1x^2 & \ldots & x^1x^{n+1} \\
(x^2)^2 & x^2x^1 & \ldots & x^2x^{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
x^{n+1}x^1 & x^{n+1}x^2 & \ldots & (x^{n+1})^2
\end{pmatrix}$$

is an imbedding.
Chapter 3

Differential equations

3.1 Vector fields and differential equations

Earlier, we mentioned that a smooth vector field on a manifold corresponds to a system of ordinary differential equations on the manifold. Indeed, recall from § 1.6 that a smooth curve $\gamma : (a, b) \to M$ is said to be an integral curve for a vector field $X$ on $M$ if

$$\gamma'(t) = X(\gamma(t)), \quad \text{for } t \in (a, b).$$

In terms of local coordinates $(U, \phi) = (U, (x^1, \ldots, x^n))$, we can write

$$X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i},$$

and then

$$\gamma'(t) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(t) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)} \quad \text{and} \quad X(\gamma(t)) = \sum_{i=1}^{n} f^i(\gamma(t)) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t)},$$

so the condition that $\gamma$ be an integral curve becomes

$$\frac{d(x^i \circ \gamma)}{dt}(t) = f^i(\gamma(t)), \quad 1 \leq i \leq n,$$

or

$$\frac{d(x^i \circ \gamma)}{dt}(t) = \bar{f}^i(x^1 \circ \gamma(t), \ldots, x^n \circ \gamma(t)), \quad 1 \leq i \leq n, \quad (3.1)$$

where $\bar{f}^i$ is the function on $\phi(U)$ defined by $\bar{f}^i = f^i \circ \phi^{-1}$. We will let $\alpha$ denote the corresponding curve $\phi \circ \gamma : (a, b) \to \mathbb{R}^n$, which is a solution to

$$\frac{d\alpha}{dt}(t) = F(\alpha(t)), \quad \text{where} \quad F : \phi(U) \to \mathbb{R}^n \quad (3.2)$$
is the $n$-tuple having component functions $(\bar{f}^1, \ldots, \bar{f}^n)$.

**Theorem 3.1.1. (Fundamental Existence and Uniqueness Theorem)** If $X$ is a smooth vector field on $M$ there is an open neighborhood $V$ of any point in $M$ and an $\delta > 0$ such that if $p \in V$, the initial value problem

$$\gamma'(t) = X(\gamma(t)), \quad \gamma(0) = p$$

(3.3)

has a unique solution $\gamma_p : (-\delta, \delta) \to M$ for $p \in V$.

This is proven, of course, by showing that the corresponding system of ODE’s in $\mathbb{R}^n$ has a unique solution. Remarkably, the proof is similar in outline to that of the Inverse Function Theorem: first one applies the Contraction Mapping Lemma to get a continuous solution to (3.2), then a bootstrapping procedure to show that the solution is smooth.

**Sketch of proof of Theorem 3.1.1:** We prove this only for a single initial value problem, the initial condition being given by a point $p$ in $M$ which lies in the domain $U$ of a smooth chart $(U, \phi)$ as described above. (A proof in the general case can be found in [10] or in books on differential equations, such as [7].) If $\phi(p) = x_0 \in \mathbb{R}^n$, we need to solve the initial value problem

$$\frac{d\alpha}{dt}(t) = F(\alpha(t)), \quad \alpha(0) = x_0.$$

But this is equivalent to the corresponding integral equation

$$\alpha(t) = x_0 + \int_0^t F(\alpha(s))ds.$$  

(3.4)

Our first goal is to construct a contraction mapping which has the solution to this integral equation as fixed point. Choose $\varepsilon > 0$ so small that $\overline{N(x_0; \varepsilon)} \subseteq \phi(U)$. Let

$$L = \sup \{\|F(x)\| : x \in \overline{N(x_0; \varepsilon)}\},$$

$$K = n \sup \left\{ \left\| \frac{\partial F}{\partial \bar{x}^i}(x) \right\| : x \in \overline{N(x_0; \varepsilon)}, 1 \leq i \leq n \right\},$$

and choose $\delta > 0$ so that

$$\delta L \leq \varepsilon, \quad 2\delta K \leq 1.$$  

(3.5)

We claim that

$$x, y \in \overline{N(x_0; \varepsilon)} \Rightarrow \|F(x) - F(y)\| \leq K\|x - y\|.$$  

Indeed,

$$\|F(x) - F(y)\| = \left\| \int_0^1 \frac{d}{dt}(F(tx + (1-t)y))dt \right\|$$

$$\leq \int_0^1 \sum_{i=1}^n \left\| \frac{\partial F}{\partial \bar{x}^i}(tx + (1-t)y) \right\| \|\bar{x}^i - \bar{y}^i\|dt \leq K\|x - y\|.$$
Next we define
\[ X = \{ \alpha : [\alpha : [-\delta, \delta] \to N(x_0; \varepsilon) \text{ continuous, with } \alpha(0) = x_0 \} \]
and define a distance function
\[ d : X \times X \to \mathbb{R} \text{ by } d(\alpha, \beta) = \sup\{ \|\alpha(t) - \beta(t)\| : t \in [-\delta, \delta] \}. \]
It is easily checked that \((X, d)\) is a metric space. Moreover, if \(\{\alpha_i\}\) is a Cauchy sequence in \((X, d)\), then for each \(t \in [-\delta, \delta]\), \(\{\alpha_i(t)\}\) is a Cauchy sequence in \(N(x_0; \varepsilon)\), and hence converges to a limit \(\alpha_\infty(t) \in N(x_0; \varepsilon)\). This defined a function \(\alpha_\infty : [-\delta, \delta] \to N(x_0; \varepsilon)\), and use of the so-called \(\varepsilon/3\)-trick (as in Chapter 7 of [16]) shows that \(\alpha_\infty\) is continuous, hence an element of \(X\), and that \(\alpha_\infty = \lim \alpha_i\). This shows that \((X, d)\) is a complete metric space.

We now use the integral equation (3.4) to define a map \(T : X \to X\) by
\[ T(\alpha)(t) = x_0 + \int_0^t F(\alpha(s)) ds. \]
Of course, we need to check that \(T(\alpha)\) is actually an element of \(X\). Since \(T(\alpha)(0) = x_0\), we need only check that
\[ T(\alpha)([\alpha, 0]) \subseteq N(x_0; \varepsilon). \]
But if \(t \in [0, \delta]\),
\[ \|T(\alpha)(t) - x_0\| = \left\| \int_0^t F(\alpha(s)) ds \right\| \leq \int_0^t \|F(\alpha(t))\| ds \leq tL \leq \delta L \leq \varepsilon, \]
the last step following from (3.5) A similar argument treats the case \(t \in [-\delta, 0]\).

Finally, we need to check that \(T\) is a contraction mapping, that is, that
\[ d(T\alpha, T\beta) \leq \frac{1}{2} d(\alpha, \beta). \quad (3.6) \]
But if \(t \in [0, \delta]\),
\[ \|T(\alpha)(t) - T(\beta)(t)\| = \left\| \int_0^t [F(\alpha(s)) - F(\beta(s))] ds \right\| \]
\[ \leq \int_0^t \|F(\alpha(s)) - F(\beta(s))\| ds \leq \int_0^t K\|\alpha(s) - \beta(s)\| ds \]
\[ \leq \int_0^t Kd(\alpha, \beta) ds \leq \delta Kd(\alpha, \beta) \leq \frac{1}{2} d(\alpha, \beta), \]
the last step following from (3.5) once again. A similar argument treats the case where \(t \in [-\delta, 0]\), so (3.6) follows.

Now we use the Contraction Mapping Lemma to conclude that the mapping \(T\) has a unique fixed point within \((X, d)\):
\[ \alpha(t) = x_0 + \int_0^t F(\alpha(s)) ds. \quad (3.7) \]
At first we only know that $\alpha$ is continuous, but then the integrand on the right-hand side of (3.4) is $C^0$, so the left-hand side is $C^1$. In the inductive step, the integrand on the right-hand side being $C^k$ implies that the left-hand side is $C^{k+1}$. By “bootstrapping,” we see that the solution is in fact $C^\infty$.

We say that an integral curve $\gamma : (a, b) \to M$ for $X$ is maximal if it is not the restriction of an integral curve $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to M$ with $(a, b)$ properly contained in $(\tilde{a}, \tilde{b})$. Local uniqueness together with Theorem 3.1.1 implies that any integral curve can be extended to a unique maximal integral curve, and this integral curve is smooth.

**Definition.** A smooth vector field $X$ on a smooth manifold $M$ is said to be complete if for each $p \in M$, the maximal integral curve $\gamma_p$ which satisfies the initial value problem

$$\gamma'(t) = X(\gamma(t)), \quad \gamma(0) = p$$

is defined on the entire real line.

If $X$ is a complete vector field on $M$, we can define a map $\phi_t : M \to M$ for each $t \in \mathbb{R}$ by $\phi_t(p) = \gamma_p(t)$, and we can define

$$\Phi : \mathbb{R} \times M \to M \quad \text{by} \quad \Phi(t, p) = \phi_t(p). \quad (3.8)$$

If $X$ is not complete, each $\phi_t$ is only locally defined and $\Phi$ is only defined on some open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$.

**Theorem 3.1.2. (Smoothness of Dependence on Initial Conditions)** If $X$ is a smooth vector field on $M$ there is an open neighborhood $U$ of any point in $M$ and a $\delta > 0$ such that the map

$$\Phi : (-\delta, \delta) \times U \to M, \quad \text{such that} \quad \Phi(t, p) = \gamma_p(t),$$

is defined on all of $(-\delta, \delta) \times U$ and is smooth.

Once we have the proof of Theorem 3.1.1, the proof of this theorem is more routine, so we refer the reader to the relevant sections of Lee [10], or to other texts on the theory of differential equations, for the argument. Theorems 3.1.1 and 3.1.2 are key results treated in basic courses on the theory of ordinary differential equations.

In the case where $X$ is complete, Theorem 3.1.2 implies that the global map $\Phi$ defined by (3.8) is smooth.

**Lemma 3.1.3.** We have $\phi_t \circ \phi_s = \phi_{t+s}$ whenever both sides are defined.

The proof is easy: the two curves

$$t \mapsto \phi_t(\phi_s(p)) \quad \text{and} \quad t \mapsto \phi_{t+s}(p)$$

are both integral curves for $X$ and both take zero to $\phi_s(p)$, so they must agree by uniqueness of solutions to initial value problems.
Thus if \( X \) is complete, Lemma 3.1.3 shows that each \( \phi_t \) is a diffeomorphism, with inverse \( \phi_{-t} \), and the map \( t \mapsto \phi_t \) is a group homomorphism from \( \mathbb{R} \) into the group \( \text{Diff}(M) \) of diffeomorphisms of \( M \), with composition as the group operation in \( \text{Diff}(M) \). We say that \( \{ \phi_t : t \in \mathbb{R} \} \) is the one-parameter group of diffeomorphisms which corresponds to \( X \). For a general vector field \( X \) which is not complete, we only get a “local one-parameter group of diffeomorphisms” of \( M \), but we will still denote it by \( \{ \phi_t : t \in \mathbb{R} \} \). We can also think of \( \{ \phi_t : t \in \mathbb{R} \} \) as the “steady-state fluid flow” determined by \( X \).

**Example.** On \( \mathbb{R}^2 \) with usual Euclidean coordinates \((x, y)\) we can consider the vector field 
\[
X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.
\]
The corresponding system of differential equations is
\[
\begin{align*}
(x \circ \gamma)'(t) &= -(y \circ \gamma)(t), \\
(y \circ \gamma)'(t) &= (x \circ \gamma)(t).
\end{align*}
\]
and it has the general solution
\[
\gamma(x, y)(t) = \phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]
so \( \phi_t \) is a counterclockwise rotation through \( t \) radians. Note that this vector field \( X \) is complete. However, the restriction of this vector field to a proper open subset \( U \subseteq \mathbb{R}^2 \) is usually not complete.

**Theorem 3.1.4.** If \( M \) is a compact smooth manifold, any vector field on \( M \) is complete.

Indeed, suppose that \( X \) is a vector field on \( M \). We first note that by Theorem 3.1.1, there exists an open cover \( \{U_\alpha : \alpha \in A\} \) and numbers \( \delta_\alpha > 0 \) such that for each \( p \in U_\alpha \), there is a solution \( \gamma_p : (-\delta_\alpha, \delta_\alpha) \to M \) to the initial value problem
\[
\gamma'_p(t) = X(\gamma(t)), \quad \gamma_p(0) = p. \tag{3.9}
\]
Since \( M \) is compact, this open cover has a finite subcover \( \{U_1, \ldots, U_m\} \) with corresponding \( \{\delta_1, \ldots, \delta_m\} \). Let \( \delta = \min(\delta_1, \ldots, \delta_m) \). Then for each \( p \in M \), there is a corresponding solution \( \gamma_p : (-\delta, \delta) \to M \) to the initial value problem (3.9). In other words, \( \phi_t : M \to M \) is a globally defined diffeomorphism when \( t \in (-\delta, \delta) \).

If \( N \in \mathbb{N} \) and \( t \in (-N\delta, N\delta) \), we can define a diffeomorphism
\[
\tilde{\phi}_t : M \to M \quad \text{by} \quad \tilde{\phi}_t = \phi_{t/N} \circ \cdots \circ \phi_{t/N}.
\]
Note that \( t \mapsto \tilde{\gamma}_p(t) = \tilde{\phi}_t(p) \) is a solution to the initial value problem (3.9) which is defined on \( (-N\delta, N\delta) \), and hence \( \tilde{\phi}_t = \phi_t \) when both are defined. Since \( N \) is arbitrary, we can in fact define \( \phi_t \) for all \( t \in \mathbb{R} \) and hence \( X \) is complete.
3.2 Lie bracket

We let $\mathcal{X}(M)$ denote the real vector space of smooth vector fields on $M$. If $X, Y \in \mathcal{X}(M)$, we define their Lie bracket $[X, Y] \in \mathcal{X}(M)$ by defining the corresponding directional derivative operator $[X, Y](p)$ at each point $p \in M$:

$$[X, Y](p)(f) = X(p)(Y(f)) - Y(p)(X(f)).$$

Using the axioms of tangent vector, we can easily verify that $[X, Y](p)$ is indeed an element of $T_pM$.

But it is perhaps easier to verify this by establishing a local coordinate formula, which also shows directly that $[X, Y]$ is smooth. If

$$X = \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{j=1}^{n} g^j \frac{\partial}{\partial x^j},$$

then

$$[X, Y](h) = X(Y(h)) - Y(X(h))$$

$$= \left( \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i} \right) \left( \sum_{j=1}^{n} g^j \frac{\partial}{\partial x^j} \right)(h) - \left( \sum_{j=1}^{n} g^j \frac{\partial}{\partial x^j} \right) \left( \sum_{i=1}^{n} f^i \frac{\partial}{\partial x^i} \right)(h)$$

$$= \sum_{j=1}^{n} \left( \sum_{i=1}^{n} f^i \frac{\partial g^j}{\partial x^i} - g^j \frac{\partial f^i}{\partial x^i} \right) \frac{\partial}{\partial x^j}(h),$$

since the second-order derivatives of $h$ cancel, so

$$[X, Y] = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} f^i \frac{\partial g^j}{\partial x^i} - g^j \frac{\partial f^i}{\partial x^i} \right) \frac{\partial}{\partial x^j}. \quad (3.10)$$

One can easily check the following properties of the Lie bracket: If $X, Y, Z \in \mathcal{X}(M)$ and $a, b \in \mathbb{R}$,

1. $[aX + bY, Z] = a[X, Z] + b[Y, Z],$
2. $[X, Y] = -[Y, X],$

The last of these is called the Jacobi identity. A real vector space $\mathfrak{g}$ together with a multiplication

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

which satisfies the above three identities is called a Lie algebra (over the real numbers $\mathbb{R}$). Thus we can say that the space $\mathcal{X}(M)$ of vector fields on a smooth manifold $M$ forms an infinite-dimensional Lie algebra. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras over $\mathbb{R}$ and

$$h : \mathfrak{g} \rightarrow \mathfrak{h}$$

is a linear map such that $h([X, Y]) = [h(X), h(Y)],$
we say that $h$ is a \textit{Lie algebra homomorphism}.

Recall that although one-forms can be pulled back under smooth maps, vector fields cannot always be pushed forward. Suppose that $F : M \to N$ is a smooth map; we say that a vector field $X$ on $M$ is $F$-related to a vector field $\tilde{X}$ on $N$ if

$$(F_*)_p(X(p)) = \tilde{X}(F(p)), \quad \text{for } p \in M.$$ 

note that $X$ is $F$-related to $\tilde{X}$ if and only if for every $f \in \mathcal{F}(F(p))$,

$$X(f \circ F) = \tilde{X}(f) \circ F.$$ 

To see this, note that

$$(F_*)_p(X(p))(f) = \tilde{X}(F(p))(f) \iff X(p)(f \circ F) = \tilde{X}(F(p))(f) \iff X(f \circ F)(p) = (\tilde{X} \circ F)(p).$$

**Proposition 3.2.1.** If $X$ and $Y$ are $F$-related to $\tilde{X}$ and $\tilde{Y}$, respectively, then $[X,Y]$ is $F$-related to $[\tilde{X}, \tilde{Y}]$.

To prove this, simply note that if $f \in \mathcal{F}(F(p))$,

$$[X,Y](f \circ F) = X(Y(f \circ F)) - Y(X(f \circ F))$$

$$= \left(\tilde{X}(\tilde{Y}(f) - \tilde{Y}(\tilde{X}(f))\right) \circ F = \left([\tilde{X}, \tilde{Y}](f)\right) \circ F.$$

**Special case:** If $F : M \to N$ is a diffeomorphism and $X$ is a vector field on $M$, we can define an $F$-related vector field $\tilde{X} = F_*(X)$ on $N$ by

$$\tilde{X}(q) = F_*(X)(q) = (F_*)_{F^{-1}(q)}(X(F^{-1}(q))), \quad \text{for } q \in N.$$ 

In this case Proposition 3.2.1 states that $F_* : \mathcal{X}(M) \to \mathcal{X}(N)$ is a Lie algebra homomorphism.

**Important fact:** If $\gamma : (a,b) \to M$ is an integral curve for $X$, then $F \circ \gamma$ is an integral curve of $\tilde{X} = F_*(X)$.

Suppose now that $\{ \phi_t : t \in \mathbb{R} \}$ is a one-parameter group of diffeomorphisms corresponding to a complete vector field $X$. If $Y$ is a smooth vector field on $M$, we can then define the $\phi_t$-related vector field $(\phi_t)_*(Y)$. If $X$ is not complete, we can still define $(\phi_t)_*(Y)(p)$ for small values of $t$:

$$(\phi_t)_*(Y)(p) = (\phi_t)_{\phi_{-t}(p)}(Y(\phi_{-t}(p))), \quad (\phi_{-t})_* (Y)(p) = (\phi_{-t})_{\phi_{t}(p)}(Y(\phi_{t}(p))).$$

Thus we get a smooth curve

$$t \mapsto (\phi_{-t})_* (Y)(p) \in T_p M.$$
Definition. The *Lie derivative* of $Y$ in the direction of $X$ is the vector field $L_X Y$ defined by

$$L_X Y(p) = \left. \frac{d}{dt} (\phi_t)_*(Y)(p) \right|_{t=0} = -\left. \frac{d}{dt} (\phi_t)_*(Y)(p) \right|_{t=0},$$

where $\{\phi_t : t \in \mathbb{R}\}$ is a local one-parameter group of diffeomorphisms corresponding to $X$.

The definition says that the Lie derivative $L_X Y$ corresponds to differentiating the vector field $Y$ along the flow $\{\phi_t\}$ determined by $X$. The way to compute it is to simply take the Lie bracket:

**Theorem 3.2.2.** $L_X Y(p) = [X,Y](p)$, for $p \in M$.

The proof is based upon the following

**Lemma 3.2.3.** If $X$ is a smooth vector field on $M$ with $X(p) \neq 0$, there is a smooth coordinate system $(U,(x^1,\ldots,x^n))$ on $M$ with $p \in U$ such that $X = \partial/\partial x^n$ on $U$.

Sketch of proof of Lemma 3.2.3: First we choose a coordinate system $\psi = (y^1,\ldots,y^n)$ on an open neighborhood $V$ of $p$ such that $\psi(p) = 0$ and $X(p) = (\partial/\partial y^n)(p)$. We then define a smooth map $F$ from a small neighborhood $W$ of $0$ in $\mathbb{R}^n$ into $M$ by

$$F(y^1,\ldots,y^{n-1},t) = \phi_t^{-1}(\psi^{-1}(y^1,\ldots,y^{n-1},0)),$$

where $\{\phi_t : t \in \mathbb{R}\}$ is a local one-parameter group of diffeomorphisms corresponding to $X$. Then $F$ is an immersion at $(0,0,\ldots,0)$, and hence by the inverse function theorem possesses a local inverse $G : U \rightarrow \mathbb{R}^n$ where $U$ is an open subset of $V$ containing $p$. We let $(x^1,\ldots,x^{n-1},x^n)$ be the components of $G$, defining a coordinate system on $U$ such that $x^n = t$. In terms of these coordinates, the local one-parameter group for $X$ is just

$$\tilde{\phi}_t(x^1,\ldots,x^{n-1},x^n) = (x^1,\ldots,x^{n-1},x^n + t).$$

In other words, in terms of these coordinates $X = \partial/\partial x^n$, which implies that $(U,G)$ is the required coordinate system.

Proof of Theorem 3.2.2: We first treat the special case where $X(p) \neq 0$. In this case we can use the Lemma to construct local coordinates $(x^1,\ldots,x^{n-1},x^n)$ so that $X = \partial/\partial x^n$. If we assume that

$$Y = \sum_{j=1}^{n} g^j \frac{\partial}{\partial x^j},$$

it then follows from (3.10) that

$$[X,Y] = \sum_{j=1}^{n} \frac{\partial g^j}{\partial x^i} \frac{\partial}{\partial x^j}. \quad (3.12)$$

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On the other hand, the formula \( X = \partial / \partial x^n \) makes it easy to solve for the local flow \( \{ \phi_t : t \in \mathbb{R} \} \) for \( X \). The corresponding system of ordinary differential equations is
\[
\frac{dx^1}{dt} = 0, \ldots, \frac{dx^{n-1}}{dt} = 0, \quad \frac{dx^n}{dt} = 1,
\]
so we obtain
\[
x^1 \circ \phi_t = x^1, \ldots, x^{n-1} \circ \phi_t = x^{n-1}, \quad x^n \circ \phi_t = x^n + t.
\]
Thus
\[
(\phi_t \ast (Y))(x^i) \circ \phi_t = Y(x^i \circ \phi_t) = \begin{cases} Y(x^i) = g^i, & \text{if } 1 \leq i \leq n - 1, \\ Y(x^i + t) = Y(x^i) = g^i, & \text{if } i = n, \end{cases}
\]
since the components \( g^i \) of \( Y \) are given by \( g^i = Y(x^i) \). Hence it follows from
\[
\phi_t \ast (Y)(p) = \sum_{i=1}^n \phi_t \ast (Y)(x^i)(p) \frac{\partial}{\partial x^i} \bigg|_p = \sum_{i=1}^n g^i \circ \phi_t^{-1}(p) \frac{\partial}{\partial x^i} \bigg|_p
\]
that
\[
\frac{d}{dt}(\phi_t \ast (Y))(p) \bigg|_{t=0} = \sum_{i=1}^n \frac{d}{dt}(g^i \circ \phi_t^{-1})(p) \bigg|_{t=0} \frac{\partial}{\partial x^i} \bigg|_p = -\sum_{i=1}^n \frac{\partial g^i}{\partial x^n}(p) \frac{\partial}{\partial x^i} \bigg|_p.
\]
Comparing with (3.12), we see that
\[
L_X Y(p) = -\frac{d}{dt}(\phi_t \ast (Y))(p) \bigg|_{t=0} = [X, Y](p),
\]
establishing the theorem when \( X(p) \neq 0 \).

But if \( X \) is identically zero in a neighborhood of \( p \), then \( L_X Y(p) \) and \([X, Y](p)\) are both zero, while if
\[
p \in \text{supp}(X) = \text{closure of } \{ p \in M : X(p) \neq 0 \},
\]
then \( L_X Y(p) = [X, Y](p) \) by continuity. QED

Corollary 3.2.4. Suppose that \( X \) and \( Y \) are vector fields on \( M \) with local one-parameter groups \( \{ \phi_t : t \in \mathbb{R} \} \) and \( \{ \psi_s : s \in \mathbb{R} \} \) respectively. Then
\[
[X, Y] = 0 \iff \phi_t \circ \psi_s = \psi_s \circ \phi_t,
\]
for all \( s \) and \( t \) sufficiently small.

Proof: If
\[
\phi_t \circ \psi_s = \psi_s \circ \phi_t,
\]
when \( s \) and \( t \) are sufficiently small, then since \( \phi_t \) takes integral curves for \( Y \) to integral curves for \( Y \), \( (\phi_t)_*(Y) = Y \) for all small \( t \), and hence

\[
[X, Y] = - \frac{d}{dt}(\phi_t)_*(Y) \bigg|_{t=0} = 0.
\]

On the other hand, since \( (\phi_t)_*(X) = X \) for all \( t \), it follows from Proposition 1.12.1 that

\[
[X, Y] = 0 \quad \Rightarrow \quad [X, (\phi_t)_*(Y)] = 0
\]

\[
\Rightarrow \quad \frac{d}{dt}(\phi_t)_*(Y)(p) \bigg|_{t=\tau} = 0 \quad \text{for all small } \tau \quad \Rightarrow \quad (\phi_t)_*(Y)(p) = Y(p)
\]

for any \( p \in M \) and all small \( t \). But this implies that \( \phi_t \) takes integral curves for \( Y \) to integral curves for \( Y \), which in turn implies that

\[
\phi_t \circ \psi_s = \psi_s \circ \phi_t,
\]

for small \( s \) and \( t \). QED

**Remark.** Suppose that \( X \) and \( Y \) are complete vector fields on \( M \) with one-parameter groups \( \{\phi_t : t \in \mathbb{R}\} \) and \( \{\psi_s : s \in \mathbb{R}\} \) respectively, such that \( [X, Y] = 0 \). Then the preceding corollary implies that the map

\[
F : \mathbb{R}^2 \rightarrow \text{Diff}(M) \quad \text{defined by} \quad F(s, t) = \phi_t \circ \psi_s = \psi_s \circ \phi_t
\]

is a group homomorphism. More generally, \( k \) commuting complete vector fields \( X_1, \ldots, X_k \) would define a group homomorphism

\[
F : \mathbb{R}^k \rightarrow \text{Diff}(M).
\]

In technical terms, this defines an *action* of the abelian Lie group \( \mathbb{R}^k \) on \( M \).

### 3.3 Lie groups revisited

Recall that if \( G \) is a Lie group and \( \sigma \in G \), we can define the *left translation*

\[
L_{\sigma} : G \rightarrow G \quad \text{by} \quad L_{\sigma}(\tau) = \sigma \tau,
\]

a map which is clearly a diffeomorphism, with inverse \( L_{\sigma^{-1}} \). Similarly, we can define *right translation*

\[
R_{\sigma} : G \rightarrow G \quad \text{by} \quad R_{\sigma}(\tau) = \tau \sigma,
\]

which is also a diffeomorphism. A vector field \( X \) on \( G \) is said to be *left invariant* if \( (L_{\sigma})_*(X) = X \) for all \( \sigma \in G \), where \( (L_{\sigma})_* \) is defined by

\[
(L_{\sigma})_*(X)(f) = X(f \circ L_{\sigma}) \circ L_{\sigma^{-1}}.
\]
If $X$ and $Y$ are left invariant vector fields on $G$, then it follows from Proposition 3.2.1 that

$$(L_\sigma)_*([X,Y]) = [(L_\sigma)_*(X),(L_\sigma)_*(Y)] = [X,Y], \quad \text{for all } \sigma \in G,$$

so their bracket $[X,Y]$ is also left invariant.

It follows that $(L_\sigma)_*$ is a Lie algebra homomorphism and the space

$$g = \{ X \in A(G) : (L_\sigma)_*(X) = X \text{ for all } \sigma \in G \}$$

is closed under Lie bracket. The real bilinear map

$$[,] : g \times g \to g$$

is skew-symmetric (that is, $[X,Y] = -[Y,X]$), and satisfies the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,$$

so $g$ is a finite-dimensional Lie algebra, and we call it the Lie algebra of the Lie group $G$. If $e$ is the identity of the Lie group, restriction to $T_eG$ yields a vector space isomorphism $\alpha : g \to T_eG$, the inverse $\beta : T_eG \to g$ being defined by $\beta(v)(\sigma) = (L_\sigma)_*(v)$. Thus the Lie algebra $g$ of a Lie group $G$ has the same dimension as $G$.

Sometimes it is also useful to consider the Lie algebra

$$\hat{g} = \{ X \in A(G) : (R_\sigma)_*(X) = X \text{ for all } \sigma \in G \}$$

of right invariant vector fields. (As an easy exercise, the reader can check that the brackets in $g$ and $\hat{g}$ have opposite sign, or that the map $X \mapsto -X$ takes left invariant vector fields to right invariant vector fields and gives an isomorphism from $g$ to $\hat{g}$.) It has become customary to use left invariant vector fields when formulating the theory of Lie algebras, but right invariant vector fields could also be used.

The most important examples of Lie groups are the general linear group

$$GL(n,\mathbb{R}) = \{ n \times n \text{ matrices } A \text{ with real entries } : \det A \neq 0 \},$$

and its subgroups. For $1 \leq i, j \leq n$, we can define coordinates

$$x^i_j : GL(n,\mathbb{R}) \to \mathbb{R} \quad \text{by} \quad x^i_j((a^i_j)) = a^i_j.$$

Of course, these are just the rectangular cartesian coordinates on an ambient Euclidean space in which $GL(n,\mathbb{R})$ sits as an open subset. If

$$X = (x^i_j) \in GL(n,\mathbb{R}) \subseteq \text{Mat}(n,\mathbb{R}) \cong \mathbb{R}^{n^2},$$

left translation by $X$ is a linear map of the ambient Euclidean space $\mathbb{R}^{n^2}$, so is its own derivative. This means that the derivative $(L_X)_*$ of left translation by $X = (x^i_j)$ takes the matrix

$$(a^i_j) \quad \text{to} \quad \left( \sum_{k=1}^{n} x^i_k a^k_j \right),$$
which implies that
\[
\beta \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_{ij}} \right) (X) = (L_X)_* \left( \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_{ij}} \right) = \sum_{i,j,k=1}^{n} x_{k}^{i} a_{jk} \frac{\partial}{\partial x_{ij}} \bigg|_{X},
\]
where \( I \) denotes the identity matrix, the identity of the Lie group \( GL(n, \mathbb{R}) \). If we allow \( X \) to vary over \( GL(n, \mathbb{R}) \), we thereby obtain a left invariant vector field
\[
X_A = \sum_{i,j,k=1}^{n} a_{ij} x_{k}^{i} \frac{\partial}{\partial x_{ij}}
\]
which is defined on \( GL(n, \mathbb{R}) \). It is the unique left invariant vector field on \( GL(n, \mathbb{R}) \) which satisfies the condition
\[
X_A(I) = \sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_{ij}} \bigg|_{I}.
\]
Every left invariant vector field on \( GL(n, \mathbb{R}) \) is obtained in this way, for some choice of \( n \times n \) matrix \( A = (a_{ij}) \).

We claim that
\[
[X_A, X_B] = X_{[A,B]}, \quad \text{where} \quad [A, B] = AB - BA.
\]
Indeed, if
\[
X^i_j = \sum_{k=1}^{n} x^k_i \frac{\partial}{\partial x^k_j},
\]
then
\[
[X^i_j, X^k_l] = \left[ \sum_{r=1}^{n} x^r_i \frac{\partial}{\partial x^r_j}, \sum_{s=1}^{n} x^s_k \frac{\partial}{\partial x^s_l} \right] = \sum_{r,s=1}^{n} x^r_i \delta^k_j \delta^s_l \frac{\partial}{\partial x^s_l} - \sum_{r,s=1}^{n} x^s_k \delta^r_i \delta^l_j \frac{\partial}{\partial x^r_i} = \delta^k_i \delta^l_j X^i_j - \delta^l_i \delta^k_j X^k_l.
\]
Thus if \( X_A = \sum_{i,j=1}^{n} a_{ij} X^i_j \) and \( X_B = \sum_{k,l=1}^{n} b_{k}^{l} X^i_l \), then
\[
[X_A, X_B] = \sum_{i,j,k,l=1}^{n} a_{ij} b_{k}^{l} [X^i_j, X^k_l] = \sum_{i,j,k,l=1}^{n} a_{ij} b_{k}^{l} X^i_l - \sum_{i,j,k,l=1}^{n} a_{ij} b_{k}^{l} X^i_k
\]
= \sum_{i,j,k,l=1}^{n} (a_{ij} b_{k}^{l} - b_{j}^{l} a_{ij}) X^i_l = X_{[A,B]},
\]
It follows that the Lie algebra of \( GL(n, \mathbb{R}) \) is isomorphic to
\[
gl(n, \mathbb{R}) \cong T_I GL(n, \mathbb{R}) = \{ n \times n \text{ matrices } A \text{ with real entries } \}.
\]
with the usual commutator of matrices as Lie bracket.

For a general Lie group $G$, if $X \in \mathfrak{g}$, the integral curve $\theta_X$ for $X$ such that $\theta_X(0) = e$ satisfies the identity $\theta_X(s + t) = \theta_X(s) \cdot \theta_X(t)$ for sufficiently small $s$ and $t$. Indeed,

$$t \mapsto \theta_X(s + t) \quad \text{and} \quad t \mapsto \theta_X(s) \cdot \theta_X(t)$$

are two integral curves for $X$ which agree when $t = 0$, and hence must agree for all $t$. From this one can easily argue that $\theta_X(t)$ is defined for all $t \in \mathbb{R}$, and thus $\theta_X$ provides a Lie group homomorphism

$$\theta_X : \mathbb{R} \to G.$$  

We call $\theta_X$ the one-parameter group which corresponds to $X \in \mathfrak{g}$. Since the vector field $X$ is left invariant, the curve

$$t \mapsto L_{\theta_X(t)} = \sigma \theta_X(t) = R_{\theta_X(t)}(\sigma)$$

is the integral curve for $X$ which passes through $\sigma$ at $t = 0$, and therefore the one-parameter group $\{\phi_t : t \in \mathbb{R}\}$ of diffeomorphisms on $G$ which corresponds to $X \in \mathfrak{g}$ is given by

$$\phi_t = R_{\theta_X(t)}, \quad \text{for } t \in \mathbb{R}.$$  

Similarly, one could show that the one-parameter group $\{\hat{\phi}_t : t \in \mathbb{R}\}$ of diffeomorphisms on $G$ which corresponds to $X \in \mathfrak{g}$ is given by

$$\hat{\phi}_t = L_{\theta_X(t)}, \quad \text{for } t \in \mathbb{R}.$$  

In the case where $G = GL(n, \mathbb{R})$ the one-parameter groups are easy to describe. In this case, if $A \in \mathfrak{gl}(n, \mathbb{R})$, we claim that the corresponding one-parameter group is given by the so-called “matrix exponential”

$$\theta_A(t) = e^{tA} = I + tA + \frac{1}{2!}t^2 A^2 + \frac{1}{2!}t^3 A^3 + \cdots.$$  

Indeed, it is easy to prove directly that the power series converges for all $t \in \mathbb{R}$, and termwise differentiation shows that it defines a smooth map. The usual proof that $e^{t+s} = e^t e^s$ extends to a proof that $e^{(t+s)A} = e^{tA} e^{sA}$, so $t \mapsto e^{tA}$ is a one-parameter group. Finally, since

$$\frac{d}{dt} (e^{tA}) = Ae^{tA} = e^{tA} A$$

is tangent to $A$ at the identity.

If $G$ is a Lie subgroup of $GL(n, \mathbb{R})$, then its left invariant vector fields are defined by taking elements of $T_I G \subseteq T_I GL(n, \mathbb{R})$ and spreading them out over $G$ by left translations of $G$. Thus the left invariant vector fields on $G$ are just the restrictions of the elements of $\mathfrak{gl}(n, \mathbb{R})$ which are tangent to $G$.

We can use the one-parameter groups to determine which elements of $\mathfrak{gl}(n, \mathbb{R})$ are tangent to $G$ at $I$. Consider, for example, the orthogonal group,

$$O(n) = \{ A \in GL(n, \mathbb{R}) : A^T A = I \}.$$  

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where \((\cdot)^T\) denotes transpose, and its identity component, the *special orthogonal group*,
\[
SO(n) = \{ A \in O(n) : \det A = 1 \}.
\]
In either case, the corresponding Lie algebra is
\[
\mathfrak{o}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{R}) : e^{tA} \in O(n) \text{ for all } t \in \mathbb{R} \}.
\]
Differentiating the equation
\[
(e^{tA})^T e^{tA} = I \quad \text{yields} \quad (e^{tA})^T A^T e^{tA} + (e^{tA})^T A e^{tA} = 0,
\]
and evaluating at \(t = 0\) yields a formula for the Lie algebra of the orthogonal group,
\[
\mathfrak{o}(n) \subseteq \{ A \in \mathfrak{gl}(n, \mathbb{R}) : A^T + A = 0 \},
\]
the Lie algebra of skew-symmetric matrices. In fact, we must have
\[
\mathfrak{o}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{R}) : A^T + A = 0 \},
\]
because both sides have the same dimension, namely \((1/2)n(n-1)\). Thus, in particular, \(\mathfrak{o}(3)\) is generated by the three matrices
\[
X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
representing infinitesimal rotations around the \(x\)-, \(y\)- and \(z\)-axes, and the Lie bracket is given by
\[
\]
We see that the Lie bracket of the rotation group \(SO(3)\) is essentially the "cross product" encountered in calculus courses.

The one-parameter groups in the complex general linear group,
\[
GL(n, \mathbb{C}) = \{ n \times n \text{ matrices } A \text{ with complex entries : } \det A \neq 0 \},
\]
are also easily described in terms of the complex matrix exponential. If
\[
A \in \mathfrak{gl}(n, \mathbb{C}) = (\text{Lie algebra of } GL(n, \mathbb{C}))
\]
\[
\cong T_1 GL(n, \mathbb{C}) = \{ n \times n \text{ matrices with complex entries } \},
\]
then the one-parameter group corresponding to \(A\) is
\[
\theta_A(t) = e^{tA} = I + tA + \frac{1}{2!} t^2 A^2 + \frac{1}{3!} t^3 A^3 + \cdots,
\]
where now all the matrices are complex. The procedure described above can also be used to determine the Lie algebras of subgroups of \(GL(n, \mathbb{C})\).
An example is the unitary group
\[ U(n) = \{ A \in GL(n, \mathbb{C}) : \bar{A}^T A = I \} \].

Differentiating \((e^{t\bar{A}})^T e^{tA} = I\) and setting \(t = 0\) shows that the Lie algebra of \(U(n)\) is
\[ u(n) = \{ A \in gl(n, \mathbb{C}) : \bar{A}^T + A = 0 \}, \]
the Lie algebra of skew-Hermitian matrices.

We claim that \(SU(n) = \{ A \in U(n) : \det A = 1 \}\) has Lie algebra
\[ su(n) = \{ A \in u(n) : \text{Trace}(A) = 0 \}. \]
Indeed, this follows from the fact that for any \(A \in u(n)\),
\[ \frac{d}{dt} \left( \det(e^{tA}) \right) \bigg|_{t=0} = \text{Trace}(A). \tag{3.14} \]
To see this, we note that if \(A\) is diagonal, say
\[ A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \]
then \(\det(tA) = e^{t\lambda_1}e^{t\lambda_2}\cdots e^{t\lambda_n}\), so
\[ \frac{d}{dt} \left( \det(e^{tA}) \right) \bigg|_{t=0} = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \]
and (3.14) holds for diagonal matrices. Since both sides are invariant under similarity transformations \(A \rightarrow B^{-1}AB\), (3.14) also holds for matrices which are similar to diagonal matrices, which includes all elements of \(u(n)\). Finally, it follows from the theory of the Jordan canonical form that almost all complex matrices are similar to diagonal matrices, so by continuity, (3.14) in fact holds for all \(n \times n\) complex matrices.

Recall that the space \(\mathbb{H}\) of quaternions can be regarded as the space of complex \(2 \times 2\) matrices of the form
\[ Q = \begin{pmatrix} t + iz & x + iy \\ -x + iy & t - iz \end{pmatrix}, \tag{3.15} \]
where \((t, x, y, z) \in \mathbb{R}^4\) and \(i = \sqrt{-1}\), and that \(\mathbb{H}\) is a skew-field, that is, all of the axioms for a field hold except for commutativity of multiplication. There is also a quaternion general linear group,
\[ GL(n, \mathbb{H}) = \{ n \times n \text{ matrices } A \text{ with quaternion entries such that } A \text{ is invertible } \}; \]
the representation (3.15) showing how this can be regarded as a Lie subgroup of $GL(2n, \mathbb{C})$. Finally, we can define the compact symplectic group

$$Sp(n) = \{ A \in GL(n, \mathbb{H}) : \bar{A}^T A = I \},$$

where the bar is now defined to be quaternion conjugation of each quaternion entry of $A$. Note that $Sp(n)$ is a compact Lie subgroup of $U(2n)$, and its Lie algebra is

$$\mathfrak{sp}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{H}) : \bar{A}^T + A = 0 \},$$

where once again conjugation of $A$ is understood to be quaternion conjugation of each matrix entry.

**Proposition 3.3.1.** If $F : G \to H$ is a Lie group homomorphism, the linear map

$$F_* : \mathfrak{g} \to \mathfrak{h}, \quad \text{defined by} \quad F_*(X) = \beta([F_*(X(e))]) \quad \text{for} \quad X \in \mathfrak{g},$$

is a Lie algebra homomorphism.

Proof: Suppose that $\{ \phi_t : t \in \mathbb{R} \}$ is the one-parameter group of diffeomorphisms corresponding to $X \in \mathfrak{g}$. Then

$$\phi_t(\sigma) = \sigma \theta_X(t), \quad \text{where} \quad \theta_X : \mathbb{R} \to G$$

is the one-parameter group corresponding to $X$. Since the on-parameter group $\theta_X : \mathbb{R} \to H$ corresponding to $X = f_*(X)$ is $F(\theta_X(t))$,

$$(F(\sigma)F_*(\sigma) = F(\sigma)F_*(\theta_X(t)) = F(\sigma)\theta_X = \hat{\phi}_t(F(\sigma)),$$

where $\{ \hat{\phi}_t : t \in \mathbb{R} \}$ is the one-parameter group of diffeomorphisms corresponding to $X = f_*(X)$. Thus $X$ is $F$-related to $\hat{X} = f_*(X)$.

It now follows from Proposition 3.2.1 that $[X, Y]$ is $F$-related to $[F_*(X), F_*(Y)]$, which in turn implies that $F_*[X, Y] = [F_*(X), F_*(Y)]$. QED

Note that when $G$ and $H$ are subgroups of general linear groups,

$$F(e^{tA}) = e^{tF_*(A)}, \quad \text{for} \quad A \in \mathfrak{g},$$

a fact often useful in calculations.

Proposition 3.3.1 states that each Lie group homomorphism induces a corresponding Lie algebra homomorphism between the corresponding Lie algebras, and this gives rise to a “covariant functor” from the category of Lie groups and Lie group homomorphisms to the category of Lie algebras and Lie algebra homomorphisms. A somewhat deeper theorem shows that for any Lie algebra $\mathfrak{g}$ there is a unique simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$. For example, there is a unique simply connected Lie group corresponding to $\mathfrak{o}(n)$, and this turns out to be a double cover of $SO(n)$ called $\text{Spin}(n)$. This correspondence between Lie groups and Lie algebras often reduces problems regarding Lie groups to Lie
algebras, which are much simpler objects that can be studied by techniques of linear algebra.

A Lie algebra is said to be simple if it is nonabelian and has no nontrivial ideals. A compact Lie group is said to be simple if its Lie algebra is simple. The compact simply connected simple Lie groups were classified by Wilhelm Killing and Élie Cartan in the late nineteenth century. In addition to the series of simple Lie groups Spin\((n)\) for \(n \geq 5\), and \(SU(n)\) and \(Sp(n)\) for \(n \geq 2\), there are exactly five exceptional Lie groups. The classification of these groups is one of the primary goals of a basic course in Lie group and Lie algebra theory.

**Exercise V.** (Due Friday, November 21.) Let

\[ SU(2) = \{ A \in U(2) : \det A = 1 \}, \]

a subgroup which can be shown to be a Lie group by the techniques we used before (see Exercise IIIA).

a. Show that \( SU(2) \) is the group of unit quaternions,

\[ SU(2) = \left\{ Q = \begin{pmatrix} t + i z & x + i y \\ -x + i y & t - i z \end{pmatrix} \in \mathbb{H} : t^2 + x^2 + y^2 + z^2 = 1 \right\}, \]

and hence is diffeomorphic to the unit sphere \( S^3 \) in \( \mathbb{R}^4 \).

b. If \( A \in SU(2) = S^3 \), we can define a linear map

\[ F(A) : \mathbb{H} \rightarrow \mathbb{H} \] \text{ by } F(A)Q = AQA^{-1}. \]

Show that \( F(A)(I) = I \), so \( F(A) \) preserves the \( t \)-axis. Show that \( F(A) \) also restricts to a rotation on \( (x, y, z) \)-space, and therefore defines a group homomorphism

\[ F : SU(2) \rightarrow SO(3). \]

Hint: Use the fact that

\[ \det Q = \det(AQA^{-1}) = t^2 + x^2 + y^2 + z^2. \]

c. Show that the induced Lie algebra homomorphism \( F_* : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3) \) is an isomorphism, and that \( F(SU(2)) \) contains an open neighborhood of the identity in \( SO(3) \).

d. Show that the group homomorphism \( F \) is onto, and that its kernel is just \( \pm I \), so \( SO(3) \) is just \( S^3 \) with antipodal points identified. (This implies that \( SO(3) \) is diffeomorphic to projective space \( \mathbb{R}P^3 \).)

**Remark.** Exercise V actually shows that the “double cover” \( \text{Spin}(3) \) of \( SO(3) \) is isomorphic to \( SU(2) \) which is just the unit sphere in the space \( \mathbb{H} \) of quaternions.

**Exercise VA.** (Do not hand in.) We now consider two copies of the special unitary group \( SU_+(2) \) and \( SU_-(2) \) and for \( (A_+, A_-) \in SU_+(2) \times SU_-(2) \), we
define a linear map
\[
F(A_+, A_-) : \mathbb{H} \to \mathbb{H} \quad \text{by} \quad F(A_+, A_-)Q = A_+QA_-^{-1}.
\]
Show that \(F(A_+, A_-)\) is a linear isomorphism of
\[
\mathbb{H} = \left\{ \begin{pmatrix} t + iz & x + iy \\ -x + iy & t - iz \end{pmatrix} \right\}
\]
and preserves the quadratic form \(t^2 + x^2 + y^2 + z^2\), so we can regard \(F(A_+, A_-)\) as an element of \(O(4)\). Show that in fact \(F(A_+, A_-) \in SO(4)\) and
\[
F : SU_+(2) \times SU_-(2) \to SO(4)
\]
is a surjective lie group homomorphism with kernel \(\{(I, I), (-I, -I)\}\). (This homomorphism turns out to be important in four-manifold topology and physics in four dimensions.)

**Remark.** Exercise VA shows that the “double cover” \(\text{Spin}(4)\) of \(SO(4)\) is isomorphic to \(SU(2) \times SU(2)\).

### 3.4 Actions of Lie groups on manifolds

When Sophus Lie (1842-1899) began investigating what came to be known as the theory of Lie groups, the abstract notion of group was not yet fully formulated. Groups were thought of as collections of transformations of some space, which included the identity transformation and were closed under composition and inverses. Gradually, it was realized that different transformation groups might be merely different “actions” of the same abstract group which can act on many different spaces.

Groups of transformations figured prominently in Felix Klein’s Erlanger Programm of 1872, which was a unification of the various geometries, Euclidean, non-Euclidean, projective, and so forth, that were studied in the nineteenth century. Klein’s idea was that each branch of geometry should be characterized by a group of transformations, and the geometry itself is concerned with studying properties of geometric figures which are invariant under this group of transformations. Thus Euclidean geometry, for example, is characterized by the group of orientation-preserving Euclidean motions which acts on Euclidean space. When the group in question is “finite-dimensional” Klein’s Erlangen Programm can be formulated in modern language as the action of a Lie group on a smooth manifold.

**Definition** A left action of a Lie group \(G\) on a smooth manifold \(M\) is a smooth map
\[
\alpha : G \times M \to M
\]
such that
1. $\alpha(e, p) = p$, when $p \in M$ and $e$ is the identity element of $G$, and
2. $\alpha(\sigma, \alpha(\tau, p)) = \alpha(\sigma \tau, p)$, for all $p \in M$ and for all $\sigma, \tau \in G$.

There is a corresponding notion of a right action of $G$ on $M$, a smooth map

$$\alpha : M \times G \longrightarrow M$$

such that

1. $\alpha(p, e) = p$, when $p \in M$ and $e$ is the identity element of $G$, and
2. $\alpha(\alpha(p, \sigma), \tau) = \alpha(p, \sigma \tau)$, for all $p \in M$ and for all $\sigma, \tau \in G$.

For simplicity, we often write $\sigma p$ for $\alpha(\sigma, p)$ or $p \sigma$ for $\alpha(p, \sigma)$.

**Example 1.** Suppose that $X$ is a complete vector field on the smooth manifold $M$ with $\{\phi_t : t \in \mathbb{R}\}$ being the corresponding one-parameter group of diffeomorphisms. Then the map $\alpha : \mathbb{R} \times M \longrightarrow M$ defined by $\alpha(t, p) = \phi_t(p)$ is a left action of the Lie group $\mathbb{R}$ on $M$.

**Example 2.** Suppose that $M = \mathbb{R}^2$ and that $G$ be the set of $3 \times 3$ real matrices of the form

$$A = \begin{pmatrix} \cos \theta & -\sin \theta & b_1 \\ \sin \theta & \cos \theta & b_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in O(2)$ and $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$.

We identify an element

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

with the column vector

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

and define a left action $\alpha : G \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\alpha \left( \begin{pmatrix} \cos \theta & -\sin \theta & b_1 \\ \sin \theta & \cos \theta & b_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

We call $G$ the group of orientation-preserving Euclidean motions, each of which can be thought of as a rotation followed by a translation. Klein would say that Euclidean plane geometry is the study of properties of geometric figures in $\mathbb{R}^2$ which are invariant under this action.

**Example 3.** Suppose that $M = \mathbb{R}P^n$ and that $G = PGL(n+1, \mathbb{R})$ the quotient of $GL(n+1, \mathbb{R})$ by the normal subgroup consisting of constant multiples of $I$.

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(It is not difficult to show that $PGL(n + 1, \mathbb{R})$ is a Lie group.) If
\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n+1}
\end{pmatrix} \in \mathbb{R}^{n+1} - \{0\}, \quad \text{we let}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n+1}
\end{pmatrix}
\]
denote the corresponding point in $\mathbb{RP}^n$. We can then define a left action $\alpha : G \times M \to M$ by
\[
\alpha \left( \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1,n+1} \\
a_{21} & a_{22} & \cdots & a_{2,n+1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n+1}
\end{pmatrix}, \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n+1}
\end{pmatrix} \right) = \begin{pmatrix}
\sum a_{1,i}x_i \\
\sum a_{2,i}x_i \\
\vdots \\
\sum a_{n+1,i}x_i
\end{pmatrix}.
\]

Then projective geometry is the study of properties of geometric figures in $\mathbb{RP}^n$ which are invariant under this action.

Let $\text{Diff}(M)$ denote the space of diffeomorphisms of $M$ with composition as the group operation. Then a left action $\alpha$ of $G$ on $M$ defines a group homomorphism $A : G \to \text{Diff}(M)$ by
\[
A(\sigma)(p) = \alpha(\sigma, p).
\]

The group action is said to be effective if $A$ has zero kernel. It is sometimes convenient to think of $\text{Diff}(M)$ as an “infinite-dimensional Lie group” with $A$ being a “Lie group homomorphism,” although it is difficult to make this precise using the theory of infinite-dimensional manifolds as presented in Lang [9]. From this point of view the Lie algebra of $\text{Diff}(M)$ should be the space of all smooth vector fields on $M$,
\[
\mathcal{X}(M) = \{ \text{smooth vector fields on } M \},
\]
with the usual Lie bracket.

We can ask whether a smooth left action of $G$ on $M$ induces a corresponding map from the Lie algebra of $G$ to $\mathcal{X}(M)$. To make this work for left actions, we need to use the Lie algebra $\hat{g}$ of right invariant vector fields on $G$.

**Proposition 3.4.1.** A smooth left action $A : G \to \text{Diff}(M)$ induces a Lie algebra homomorphism
\[
A_* : \hat{g} \to \mathcal{X}(M).
\]

To prove the this, we suppose that we are given a left action $\alpha : G \times M \to M$. Recall that an element $\tilde{X} \in \hat{g}$ determines a corresponding one-parameter group $\theta_{\tilde{X}} : \mathbb{R} \to G$ and this in turn determines a one-parameter group of diffeomorphisms $\phi_t : M \to M$ by $\phi_t(p) = \alpha(\theta_{\tilde{X}}(t), p)$. We let $\tilde{X}$ denote the vector field on $M$ generated by this one-parameter group of diffeomorphisms and set $A_*(\tilde{X}) = \tilde{X}$. We need to check that $A_*$ is a Lie algebra homomorphism.
Now note that the integral curve
\[ t \mapsto (\theta_X(t)\sigma, p) \quad \text{for} \quad (\hat{X}, 0) \quad \text{on} \quad G \times M \]
is taken by the map \( \alpha \) to
\[ t \mapsto \alpha(\theta_X(t), \alpha(\sigma, p)) = \phi_t(\alpha(\sigma, p)), \]
an integral curve for \( \hat{X} \).

Thus the vector field \( (\hat{X}, 0) \) on \( G \times M \) is \( \alpha \)-related to \( A_*(\hat{X}) = \tilde{X} \).

Finally, we use Proposition 3.2.1 to show that
\( (\hat{X}, 0) \) and \( (\hat{Y}, 0) \) \( \alpha \)-related to \( \hat{X} \) and \( \hat{Y} \)
respectively implies that
\[ ([\hat{X}, \hat{Y}], 0) \quad \text{is} \quad \alpha \text{-related to} \quad [\tilde{X}, \tilde{Y}], \]
and hence that \( A_*([\hat{X}, \hat{Y}]) = [A_*(\hat{X}), A_*(\hat{Y})] \). QED

Thus we see that a left action of a Lie group \( G \) on \( M \) generates a finite-dimensional Lie subalgebra \( g_0 \subseteq \mathcal{X}(M) \) which is isomorphic to the Lie algebra of \( G \) when the action is effective. In more advanced courses, the “theorem of Frobenius” is used to show that under appropriate hypotheses a finite-dimensional lie subalgebra \( g_0 \subseteq \mathcal{X}(M) \) consisting of complete vector fields gives rise to a left action of some Lie group \( G \) on \( M \), where \( G \) is some Lie group with Lie algebra isomorphic to \( g_0 \).
Chapter 4

Differential forms

Le savant digne de ce nom, le géomètre surtout, prouve en face de son oeuvre la même impression que l’artiste; sa jouissance est aussi grande et de même nature. (Henri Poincaré, 1890)

Our next goal is to develop the technology needed to fully understand the integral theorems from calculus of several variables: Green’s Theorem, the Divergence Theorem and Stokes’ Theorem. We will give a formulation which makes these integral theorems special cases of a “generalized Stokes’ Theorem” on \( n \)-dimensional oriented smooth manifolds with boundary.

This generalized version of Stokes’s theorem was known to Henri Poincaré (1854-1912), and might have suggested the notion of homology which Poincaré investigated in his first major research article on algebraic topology, his celebrated \textit{Analysis situs} of 1895. Poincaré and Élie Joseph Cartan (1869-1951) gradually became aware that the topological invariants of smooth manifolds (the number of \( k \)-dimensional holes, also called the Betti numbers, and later called the dimensions of the real cohomology groups) could be estimated, then fully determined by integrating covariant tensor fields (differential forms of degree \( k \)) over compact oriented submanifolds. These ideas were formalized by Cartan’s student Georges de Rham in 1931. In the later parts of this chapter, we will describe the basics of the resulting de Rham cohomology theory using strongly the important fact that covariant tensor fields pull back under smooth maps. A more extended treatment of de Rham theory can be found in Bott and Tu [3].

4.1 Tensor algebra

At first we will focus on covariant tensors based upon \( T^*_p M \) instead of the contravariant tensors based upon \( T_p M \), because they are better behaved under smooth maps.

Recall that \( T^*_p M \) is the real vector space of linear maps \( \alpha : T_p M \to \mathbb{R} \). In analogy, we define the tensor product \( T^*_p M \otimes T^*_p M \) to be the vector space of
bilinear maps
\[ \phi : T_p M \times T_p M \rightarrow \mathbb{R}. \]
By bilinear we mean that for each \( w \in T_p M \), the maps
\[ v \mapsto \phi(v, w) \quad \text{and} \quad v \mapsto \phi(w, v) \]
are linear. If \( \alpha, \beta \in T_p^* M \), we define their tensor product
\[ \alpha \otimes \beta \in T_p^* M \otimes T_p^* M \quad \text{by} \quad (\alpha \otimes \beta)(v, w) = \alpha(v)\beta(w). \]
More generally, a map
\[ \phi : T_p M \times T_p M \times \cdots \times T_p M \rightarrow \mathbb{R} \]
is \( k \)-multilinear if it is linear in each variable separately when the other variables are held fixed. We let \( \otimes^k T_p^* M \) denote the vector space of \( k \)-multilinear maps
\[ \phi : \underbrace{T_p M \times T_p M \times \cdots \times T_p M}_{k} \rightarrow \mathbb{R}. \]
If \( \phi \in \otimes^k T_p^* M \) and \( \psi \in \otimes^\ell T_p^* M \), we define their tensor product
\[ \phi \otimes \psi \in \otimes^{k+\ell} T_p^* M \]
by \( (\phi \otimes \psi)(v_1, \ldots, v_{k+\ell}) = \phi(v_1, \ldots, v_k)\psi(v_{k+1}, \ldots, v_{k+\ell}) \).
This tensor product operation is bilinear,
\[ (a\phi_1 + \phi_2) \otimes \psi = a\phi_1 \otimes \psi + \phi_2 \otimes \psi, \quad \phi \otimes (a\psi_1 + \psi_2) = a\phi \otimes \psi_1 + \phi \otimes \psi_2, \]
and associative,
\[ (\phi \otimes \psi) \otimes \omega = \phi \otimes (\psi \otimes \omega). \]
Thus we can write \( \phi \otimes \psi \otimes \omega \) with no danger of confusion. In algebraic language, we would say that the covariant tensor algebra
\[ \otimes^* T_p^* M = \mathbb{R} \oplus T_p^* M \oplus (T_p^* M \otimes T_p^* M) \oplus \cdots \oplus (\otimes^k T_p^* M) \oplus \cdots \]
is an associative algebra over \( \mathbb{R} \) with unit.
If \( M \) has dimension \( n \) and \((U, (x^1, \ldots, x^n))\) is a smooth coordinate system on \( M \) with \( p \in U \), we can define
\[ dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p \in \otimes^k T_p^* M \]
by
\[ dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p \left( \sum a_1^{i_1} \frac{\partial}{\partial x^{j_1}}|_p, \ldots, \sum a_k^{i_k} \frac{\partial}{\partial x^{j_k}}|_p \right) = a_1^{i_1} \cdots a_k^{i_k} \]
Proposition 4.1.1. The vector space $\otimes^k T^*_p M$ has dimension $n^k$ with basis
\[\{dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p \in \otimes^k T^*_p M : 1 \leq i_1, \ldots, i_k \leq n\}\].

Proof: Suppose that
\[\sum a_{i_1 \cdots i_k} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p = 0.\]
Then
\[0 = \sum a_{i_1 \cdots i_k} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p \left( \frac{\partial}{\partial x^{j_1}}|_p, \ldots, \frac{\partial}{\partial x^{j_k}}|_p \right) = a_{j_1 \cdots j_k},\]
establishing linear independence. On the other hand, suppose that $\phi \in \otimes^k T^*_p M$ and set
\[a_{i_1 \cdots i_k} = \phi \left( \frac{\partial}{\partial x^{i_1}}|_p, \ldots, \frac{\partial}{\partial x^{i_k}}|_p \right).\]
We claim that
\[\phi = \sum a_{i_1 \cdots i_k} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p.\]
Indeed, if
\[v_1 = \sum b_{i_1}^{i_1} \frac{\partial}{\partial x^{i_1}}|_p, \ldots, v_k = \sum b_{i_k}^{i_k} \frac{\partial}{\partial x^{i_k}}|_p,\]
then
\[\phi(v_1, \ldots, v_k) = \sum b_{i_1}^{i_1} \cdots b_{i_k}^{i_k} \phi \left( \frac{\partial}{\partial x^{i_1}}|_p, \ldots, \frac{\partial}{\partial x^{i_k}}|_p \right) = \sum a_{i_1 \cdots i_k} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p(v_1, \ldots, v_k).\] QED

We now let $\otimes^k T^*_M$ be the disjoint union of $\otimes^k T^*_p M$ as $p$ ranges throughout $M$, and define
\[\pi : \otimes^k T^*_M \rightarrow M \text{ by } \pi(\otimes^k T^*_p M) = \{p\}.\]
We claim that $\otimes^k T^*_M$ is a smooth manifold of dimension $n + n^k$.

To see this, we let $A = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ be a smooth atlas on $M$ and let
\[\phi_\alpha = (x^1_\alpha, \ldots, x^n_\alpha)\]. We let $\tilde{U}_\alpha = \pi^{-1}(U_\alpha)$, and define
\[p_{i_1 \cdots i_k}^{\alpha} : \tilde{U}_\alpha \rightarrow \mathbb{R} \text{ by } p_{i_1 \cdots i_k}^{\alpha} \left( \sum a_{j_1 \cdots j_k} dx^{j_1}|_p \otimes \cdots \otimes dx^{j_k}|_p \right) = a_{i_1 \cdots i_k}.\]
We then take
\[\tilde{\phi}_\alpha = (x^1_\alpha \circ \pi, \ldots, x^n_\alpha \circ \pi, \ldots, p_{i_1 \cdots i_k}^{\alpha}, \cdots)\]
as a smooth coordinate system on $\tilde{U}_\alpha$. Just as in the case of the cotangent bundle, we can check that $\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}$ is smooth where defined. We give $\otimes^k T^* M$ the unique topology which makes each $\tilde{\phi}_\alpha$ into a homeomorphism from $\tilde{U}_\alpha$ to an open subset of $\mathbb{R}^{n+n^k}$, and check that this topology is Hausdorff and second countable. Then

$$\tilde{A} = \{(\tilde{U}_\alpha, \tilde{\phi}_\alpha) : \alpha \in A\}$$

is an atlas on $\otimes^k T^* M$ making it into a smooth manifold.

**Definition.** A smooth covariant tensor field of rank $k$ on $U \subseteq M$ is a smooth map

$$\phi : U \rightarrow \otimes^k T^* M$$

such that $\pi \circ \phi = \text{id}_U$.

We let $\Gamma(\otimes^k T^* M | U)$ denote the space of covariant tensor fields of rank $k$ on $U \subseteq M$.

Note that if $\phi$ and $\psi$ are smooth covariant tensor fields of ranks $k$ and $\ell$ respectively, we can define their tensor product, a smooth covariant tensor field $\phi \otimes \psi$ of rank $k + \ell$ by

$$\phi \otimes \psi (p) = \phi(p) \otimes \psi(p).$$

If $(U, (x^1, \ldots, x^n))$ is a smooth coordinate system on $M$, we can define covariant tensor fields

$$dx^{i_1} \otimes \cdots \otimes dx^{i_k} : U \rightarrow \otimes^k T^* M$$

by

$$dx^{i_1} \otimes \cdots \otimes dx^{i_k} (p) = dx^{i_1} \big|_p \otimes \cdots \otimes dx^{i_k} \big|_p.$$  

If $\phi$ is an arbitrary covariant tensor field of rank $k$ over $U$,

$$\phi = \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n f_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k},$$

where each

$$f_{i_1 \cdots i_k} : U \rightarrow \mathbb{R}$$

is a smooth function. We say that the functions $f_{i_1 \cdots i_k}$ are the components of $\phi$ with respect to the coordinates $(x^1, \ldots, x^n)$.

One could define the tensor product $\otimes^k T_p M$ as the space of $k$-multilinear real-valued maps on $T^*_p M$. One could then define $\otimes^k T M$, give it a smooth manifold structure and define contravariant tensor fields of rank $k$ in the same way we defined covariant tensor fields of rank $k$. But covariant tensor fields have the advantage that they can be pulled back under smooth maps. Indeed, a smooth map $F : M \rightarrow N$ induces a linear map

$$F^*_p : \otimes^k T_{F(p)}^* N \rightarrow \otimes^k T_p^* M$$

by

$$F^*_p (\phi)(v_1, \ldots, v_k) = \phi((F_p)_*(v_1), \ldots, (F_p)_*(v_k)),$$
which preserves the tensor product,
\[ F^*_p(\phi \otimes \psi) = F^*_p(\phi) \otimes F^*_p(\psi). \]

This in turn induces a linear map
\[ F^* : \Gamma(\otimes^k T^* N|U) \to \Gamma(\otimes^k T^* M|(F^{-1}(U))) \]
by \( (F^*(\phi))(p) = F^*_p(\phi(F(p))). \)

**Important examples of covariant tensor fields: Riemannian metrics.**

**Rough Definition.** Let \( M \) be a smooth manifold. A *Riemannian metric* on \( M \) is a function which assigns to each \( p \in M \) a (positive-definite) inner product \( \langle \cdot, \cdot \rangle_p \) on \( T_p M \) which “varies smoothly” with \( p \in M \).

Of course, we have to explain what we mean by “vary smoothly,” and this is exactly what the notion of smooth covariant tensor field does for us:

**More Precise Definition.** Let \( M \) be a smooth manifold. A *Riemannian metric* on \( M \) is a smooth covariant tensor field \( g : M \to \otimes^2 T^* M \) such that at each \( p \in M \),
\[ g(p) = \langle \cdot, \cdot \rangle_p : T_p M \times T_p M \to \mathbb{R} \]
is symmetric and positive-definite. A *Riemannian manifold* is a smooth manifold with a Riemannian metric.

If \( \phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n \) is a smooth coordinate system on a Riemannian manifold \( M \), then we can write
\[ g|U = \langle \cdot, \cdot \rangle|_U = \sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j, \]
where the component functions \( g_{ij} : U \to \mathbb{R} \) are smooth and fit together into a matrix
\[
\begin{pmatrix}
g_{11} & \cdots & g_{1n} \\
\vdots & \ddots & \vdots \\
g_{n1} & \cdots & g_{nn}
\end{pmatrix}
\]
which is symmetric and positive definite at each point \( p \in U \).

**Example 1.** The simplest example of a Riemannian manifold is \( n \)-dimensional *Euclidean space* \( \mathbb{E}^n \), which is simply \( \mathbb{R}^n \) together with its standard rectangular cartesian coordinate system \((x^1, \ldots, x^n)\), and the Euclidean metric
\[ \langle \cdot, \cdot \rangle_E = ds^2 = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n. \]
In this case, the inner product
\[ \langle \cdot, \cdot \rangle|_p : T_p \mathbb{R}^n \times T_p \mathbb{R}^n \to \mathbb{R} \]
is just the ordinary dot product, and the components of the metric are simply
\[ g_{ij} = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \]

**Example 2.** Suppose that \( M \) is an \( n \)-dimensional smooth manifold and that \( F : M \to \mathbb{R}^N \) is a smooth imbedding. We can give \( \mathbb{R}^N \) the Euclidean metric defined in the preceding example. For each choice of \( p \in M \), we can then define an inner product \( \langle \cdot, \cdot \rangle_p \) on \( T_p M \) by
\[ \langle v, w \rangle_p = \langle F_{*p}(v), F_{*p}(w) \rangle_E = F_{*p}(v) \cdot F_{*p}(w), \quad \text{for } v, w \in T_p M, \]
where the dot on the right denotes the usual Euclidean dot product. Here \( F_{*p} \) is the differential of \( F \) at \( p \) defined in terms of a smooth coordinate system \( \phi = (x^1, \ldots, x^n) \) by the explicit formula
\[ F_{*p} \left( \frac{\partial}{\partial x^i} \bigg|_p \right) = D_i(F \circ \phi^{-1})(\phi(p)) \in \mathbb{R}^N. \]
Clearly, \( \langle v, w \rangle_p \) is symmetric, and it is positive definite because \( F \) is an immersion. Moreover,
\[ g_{ij}(p) = \left\langle \frac{\partial}{\partial x^i} \bigg|_p, \frac{\partial}{\partial x^j} \bigg|_p \right\rangle = F_{*p} \left( \frac{\partial}{\partial x^i} \bigg|_p \right) \cdot F_{*p} \left( \frac{\partial}{\partial x^j} \bigg|_p \right) \]
so \( g_{ij}(p) \) depends smoothly on \( p \). This Riemannian metric on \( M \) can be described as simply the pullback via \( F^* \) of the Euclidean metric from \( \mathbb{R}^N \):
\[ \langle \cdot, \cdot \rangle = F^* \langle \cdot, \cdot \rangle_E. \]
Thus any imbedding \( F \) from \( M \) into \( \mathbb{R}^N \) induces a Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( M \) which we call the induced metric. It is an interesting fact that this construction includes all Riemannian manifolds.

**Definition.** Let \((M, \langle \cdot, \cdot \rangle)\) be a Riemannian manifold, and suppose that \( \mathbb{E}^N \) denotes \( \mathbb{R}^N \) with the Euclidean metric. An imbedding \( F : M \to \mathbb{E}^N \) is said to be isometric if \( \langle \cdot, \cdot \rangle = F^* \langle \cdot, \cdot \rangle_E \).

**Theorem 4.1.2. (Nash’s Imbedding Theorem)** If \((M, \langle \cdot, \cdot \rangle)\) is any smooth Riemannian manifold, there exists an isometric imbedding \( F : M \to \mathbb{E}^N \) into some Euclidean space.

This was regarded as a landmark achievement when its proof first appeared in 1956, and it transformed the landscape of differential geometry. However, the proof is difficult, involving subtle techniques from the theory of nonlinear partial differential equations, and unfortunately, the proof is beyond the scope of this course.
A special case of Example 2 consists of two-dimensional smooth manifolds which are imbedded in $\mathbb{R}^3$. These are usually called smooth surfaces in $\mathbb{R}^3$ and are studied extensively in undergraduate courses in “curves and surfaces.” This subject was extensively developed during the nineteenth century and was summarized in 1887-96 in a monumental four-volume work, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, by Jean Gaston Darboux. Indeed, the theory of smooth surfaces in $\mathbb{R}^3$ still provides much geometric intuition regarding Riemannian geometry of higher dimensions.

What kind of geometry does a Riemannian metric provide a smooth manifold $M$? Well, to begin with, we can use a Riemannian metric to define the lengths of tangent vectors. If $v \in T_pM$, we define the length of $v$ by the formula

$$\|v\| = \sqrt{\langle v, v \rangle_p}.$$  

Second, we can use the Riemannian metric to define angles between vectors: The angle $\theta$ between two nonzero vectors $v, w \in T_pM$ is the unique $\theta \in [0, \pi]$ such that

$$\langle v, w \rangle_p = \|v\|\|w\| \cos \theta.$$  

Third, one can use the Riemannian metric to define lengths of curves. Suppose that $\gamma : [a, b] \to M$ is a smooth curve with velocity vector

$$\gamma'(t) = \sum_{i=1}^n dx_i \frac{\partial}{\partial x_i} \bigg|_{\gamma(t)} \in T_{\gamma(t)}M, \quad \text{for } t \in [a, b].$$  

Then the length of $\gamma$ is given by the integral

$$L(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} dt.$$  

We can also write this in local coordinates as

$$L(\gamma) = \int_a^b \sqrt{\sum_{i,j=1}^n g_{ij}(\gamma(t)) \frac{dx_i}{dt} \frac{dx_j}{dt}} dt.$$  

Finally, we will see later that a Riemannian metric allows us to calculate volumes of regions within $M$.

Note that if $F : M \to \mathbb{E}^N$ is an isometric imbedding, then $L(\gamma) = L(F \circ \gamma)$. Thus the lengths of a curve on a smooth surface in $\mathbb{E}^3$ is just the length of the corresponding curve in the ambient Euclidean space $\mathbb{E}^3$.

Since any Riemannian manifold can be isometrically imbedded in some $\mathbb{E}^N$, one might be tempted to try to study the Riemannian geometry of $M$ via the Euclidean geometry of the ambient Euclidean space. However, this is not an efficient approach, since sometimes the isometric imbedding is quite difficult to construct and provides extraneous information.
Example 3. Suppose that $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, with Riemannian metric

$$\langle \cdot, \cdot \rangle = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

A celebrated theorem of David Hilbert states that $(\mathbb{H}^2, \langle \cdot, \cdot \rangle)$ has no isometric imbedding in $\mathbb{E}^3$ and although isometric imbeddings in Euclidean spaces of higher dimension can be constructed via Nash’s Theorem, none of them is easy to describe. The Riemannian manifold $(\mathbb{H}^2, \langle \cdot, \cdot \rangle)$ is called the Poincaré upper half-plane, and figures prominently in the theory of Riemann surfaces. It is the foundation for the non-Euclidean geometry discovered by Bolyai and Lobachevsky in the nineteenth century.

Open Problem 4.1.3. Is it true that if $(M, g)$ is a smooth two-dimensional Riemannian manifold and $p \in M$, there must exist an open neighborhood $U$ of $p$ in $M$ and an isometric immersion $F : U \to \mathbb{R}^3$? The answer to this question is “yes” when $(M, g)$ is real-analytic by a famous theorem due to Janet and Cartan. This theorem states that if $(M, g)$ is $n$-dimensional real analytic Riemannian manifold and $p \in M$, then there is an isometric immersion $F : U \to \mathbb{R}^{(1/2)n(n+1)}$, where $U$ is some open neighborhood of $p$. This was first proven by Cartan in 1927 as an application of his theory of exterior differential systems [4], a method for reducing nonlinear systems of PDE’s to the Cauchy-Kovaleskaya Theorem.

We will return to treat the geometry of Riemannian metrics in much more detail in 240BC.

4.2 Exterior algebra

Let $\Lambda^k T^*_p M$ denote the space of **alternating or skew-symmetric** maps

$$\phi : \overbrace{T_p M \times T_p M \times \cdots \times T_p M}^{k} \to \mathbb{R},$$

a real vector space. By alternating, we mean that if $i < j$,

$$\phi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\phi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

An alternate description can be given in terms of the symmetric group

$$S_k = \text{ bijections } \sigma \text{ from } \{1, \ldots, k\} \text{ onto itself},$$

a group under composition. We define a group homomorphism

$$\text{sgn} : S_k \to \{\pm 1\} \quad \text{by} \quad \text{sgn}(\sigma) = \Pi_{1 \leq i < j \leq k} \frac{\sigma(i) - \sigma(j)}{i - j}.$$  

Then an element $\phi \in \otimes^k T^*_p M$ is alternating if and only if

$$(\sigma \cdot \phi)(v_1, \ldots, v_k) = \phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \text{sgn}(\sigma)\phi(v_1, \ldots, v_k).$$  

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for all $\sigma \in S^k$ and for all choices of $v_1, \ldots, v_k$. We define a map

$$\text{Alt} : \otimes^k T^*_p M \longrightarrow \Lambda^k T^*_p M \quad \text{by} \quad \text{Alt}(\phi) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)(\sigma \cdot \phi),$$

or equivalently, by

$$\text{Alt}(\phi)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

It is easily checked that $\text{Alt} \circ \text{Alt} = \text{Alt}$, so $\text{Alt}$ is a projection to the linear subspace

$$\Lambda^k T^*_p M \subseteq \otimes^k T^*_p M.$$

If $\phi \in \Lambda^k T^*_p M$ and $\psi \in \Lambda^\ell T^*_p M$, we can define their wedge product

$$\phi \wedge \psi \in \Lambda^{k+\ell} T^*_p M \quad \text{by} \quad \phi \wedge \psi = \frac{(k+\ell)!}{k!\ell!} \text{Alt}(\phi \otimes \psi).$$

It should be noted that some older authors use an alternate definition

$$\phi \tilde{\wedge} \psi = \text{Alt}(\phi \otimes \psi), \quad (4.1)$$

which puts additional factors into some of the formulae we will derive later. The wedge product is bilinear,

$$(a\phi_1 + \phi_2) \wedge \psi = a\phi_1 \wedge \psi + \phi_2 \wedge \psi, \quad \phi \wedge (a\psi_1 + \psi_2) = a\phi \wedge \psi_1 + \phi \wedge \psi_2,$$

skew-commutative,

$$\phi \wedge \psi = (-1)^{k\ell} \psi \wedge \phi \quad \text{when} \quad \phi \in \Lambda^k T^*_p M \quad \text{and} \quad \psi \in \Lambda^\ell T^*_p M$$

and associative,

$$(\phi \wedge \psi) \wedge \omega = \phi \wedge (\psi \wedge \omega).$$

Only associativity is mildly difficult to check.

It is perhaps a little easier to check associativity for the alternate definition (4.1) which means to simply verify the identity

$$\text{Alt}(\phi \otimes \text{Alt}(\psi \otimes \omega)) = \text{Alt}(\phi \otimes \psi \otimes \omega) = \text{Alt}(\text{Alt}(\phi \otimes \psi) \otimes \omega). \quad (4.2)$$

To prove this identity, we suppose that

$$\phi \in \Lambda^k T^*_p M, \quad \psi \in \Lambda^\ell T^*_p M, \quad \omega \in \Lambda^m T^*_p M.$$

Then the first equality in (4.2) follows from the calculation

$$\text{Alt}(\phi \otimes \text{Alt}(\psi \otimes \omega))$$

$$= \frac{1}{(k+\ell+m)!} \sum_{\sigma \in S_{k+\ell+m}} \sigma \cdot (\text{Alt}(\phi \otimes \psi) \otimes \omega)$$

$$= \frac{1}{(k+\ell+m)!} \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell+m}} \sum_{\tau \in S_{k+\ell}} \sigma \cdot (\tau \cdot (\phi \otimes \psi) \otimes \omega)$$

$$= \frac{1}{(k+\ell+m)!} \sum_{\sigma \in S_{k+\ell+m}} \sigma \cdot (\phi \otimes \psi \otimes \omega) = \text{Alt}(\phi \otimes \psi \otimes \omega).$$
The other inequality is proven the same way.

The identity (4.2) shows that $\wedge$ is associative, so we can write $\phi \wedge \psi \wedge \omega$ with no danger of confusion. Then for the wedge product that we actually use,

\[
(\phi \wedge \psi) \wedge \omega = \frac{(k + \ell)!}{k!\ell!} (\phi \wedge \psi) \wedge \omega
\]

\[
= \frac{(k + \ell)!}{k!\ell!} \frac{(k + \ell + m)!}{(k + \ell)!m!} (\phi \wedge \psi) \wedge \omega = \frac{(k + \ell + m)!}{k!\ell!m!} \phi \wedge \psi \wedge \omega.
\]

Similarly,

\[
\phi \wedge (\psi \wedge \omega) = \frac{(k + \ell + m)!}{k!\ell!m!} \phi \wedge \psi \wedge \omega,
\]

which gives the desired associativity.

The upshot is that the covariant exterior algebra

\[
\Lambda^*T^*_pM = \mathbb{R} \oplus T^*_pM \oplus (T^*_pM \wedge T^*_pM) \oplus \cdots \oplus (\Lambda^kT^*_pM) \oplus \cdots \oplus (\Lambda^nT^*_pM)
\]

is a skew-commutative associative algebra over $\mathbb{R}$ with unit. The total dimension of this algebra as a real vector space is $2^n$, where $n$ is the dimension of $M$.

**Proposition 4.2.1.** If $(x^1, \ldots, x^n)$ is a smooth coordinate system on $U \subseteq M$, then

\[
\{dx^i|_p \wedge \cdots \wedge dx^k|_p \in \Lambda^kT^*_pM : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}
\]

is a basis for $\Lambda^kT^*_pM$; hence $\Lambda^kT^*_pM$ has dimension

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

Half of the proof is easy: Since the collection

\[
\{dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p \in \otimes^kT^*_pM : 1 \leq i_1, \ldots, i_k \leq n\}
\]

spans $\otimes^kT^*_M$,

\[
\{dx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p \in \Lambda^kT^*_pM : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}
\]

must span $\Lambda^kT^*_pM$. On the other hand, if

\[
\sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p = 0,
\]

then

\[
0 = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx^{i_1}|_p \otimes \cdots \otimes dx^{i_k}|_p \left( \frac{\partial}{\partial x^{j_1}}|_p, \ldots, \frac{\partial}{\partial x^{j_k}}|_p \right) = a_{j_1 \cdots j_k},
\]

which establishes linear independence. QED
We now let $\Lambda^k T^* M$ be the disjoint union of $\Lambda^k T^*_p M$ as $p$ ranges throughout $M$, and define

$$\pi : \Lambda^k T^* M \longrightarrow M \text{ by } \pi (\Lambda^k T^*_p M) = \{p\}.$$ 

By a slight modification of the argument in §4.1 we can show that $\Lambda^k T^* M$ is a smooth manifold, this time of dimension $n + \binom{n}{k}$. This is the total space of a “vector bundle” over $M$, as described in the opening chapters of Milnor and Stasheff’s highly recommended book [13].

**Definition.** A smooth differential form of rank $k$ on $U \subseteq M$, or a smooth $k$-form, is a smooth map $\omega : U \longrightarrow \Lambda^k T^* M$ such that $\pi \circ \omega = \text{id}_U$.

We let $\Omega^k(U)$ denote the space of smooth $k$-forms on $U \subseteq M$.

Note that if $\omega$ and $\phi$ are smooth differential forms of ranks $k$ and $\ell$ respectively, we can define a smooth differential form $\omega \wedge \phi$ of rank $k + \ell$ by

$$(\omega \wedge \phi)(p) = \omega(p) \wedge \phi(p).$$

If $(U, (x^1, \ldots, x^n))$ is a smooth coordinate system on $M$, this definition of wedge product yields a collection of smooth $k$-forms

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} : U \longrightarrow \Lambda^k T^* M$$

such that

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} (p) = dx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p.$$ 

If $\omega$ is an arbitrary smooth $k$-form over $U$, we can write

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where each

$$f_{i_1 \cdots i_k} : U \longrightarrow \mathbb{R}$$

is a smooth function. We say that the functions $f_{i_1 \cdots i_k}$ are the components of $\omega$ with respect to the coordinates $(x^1, \ldots, x^n)$.

### 4.3 The exterior derivative

The exterior derivative gives an extension of the familiar operators of vector calculus (gradient, divergence and curl) from $\mathbb{R}^3$ to $n$-dimensional smooth manifolds. Recall that if $M$ is a smooth manifold, $\Omega^k(U)$ denotes the space of smooth $k$-forms on $U \subseteq M$, with $\Omega^0(U) = F(U)$ being the space of smooth real-valued functions on $U$.

**Theorem 4.3.1.** For any smooth manifold $M$, there is a unique collection of $\mathbb{R}$-linear maps

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

which satisfy the following axioms:
1. If $f \in \Omega^0(M)$, the $d(f)$ is the differential $df$ previously defined,
2. $d \circ d = 0$, and
3. if $\omega \in \Omega^k(M)$ and $\phi \in \Omega^\ell(M)$, then
   \[ d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^k \omega \wedge d\phi. \]

The operator $d$ is called the exterior derivative.

If $\omega$ is a smooth $k$-form on $M$, the support of $\omega$
\[ \text{supp}(\omega) = \text{closure of } \{ p \in M : \omega(p) \neq 0 \}. \]

**Lemma 4.3.2.** If $d$ exists, it is “local,” which means that
\[ \text{supp}(d\omega) \subseteq \text{supp}(\omega), \text{ for } \omega \in \Omega^k(M). \]

To prove the lemma, it suffices to show that
\[ p \notin \text{supp}(\omega) \implies p \notin \text{supp}(d\omega). \]
Suppose that $\omega \equiv 0$ on an open neighborhood $V$ of $p$. Let $f : V \to [0, 1]$ be a smooth function such that $f \equiv 1$ on an open neighborhood $W$ of $p$ and $f \equiv 0$ on $M - V$. Then $f \cdot \omega \equiv 0$, so if $q \in W$,
\[ 0 = d(f \omega)(q) = df(q) \wedge \omega(q) + f(q)(d\omega)(q) = d\omega(q), \]
so $d\omega|W \equiv 0$ and $p \notin \text{supp}(d\omega)$. QED

The key point of this lemma is that it allows us to restrict any collection of real linear operators on $M$ satisfying the three axioms to a unique collection of such operators on any open subset $U \subseteq M$ by using smooth cutoff functions.

**Proof of uniqueness of $d$:** Lemma 4.3.2 implies that it suffices to establish uniqueness when $M = U$ is the domain of a local coordinate system $(x^1, \ldots, x^n)$. Note first that it follows from axioms 2 and 3 together with induction that
\[ d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = 0. \]
(4.3)

If $\omega \in \Omega^k(U)$, then
\[ \omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \]
so it follows from the axioms for $d$ that
\[
\begin{align*}
    d\omega &= d \left( \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) \\
    &= \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} d(dx^{i_1} \wedge \cdots \wedge dx^{i_k})
\end{align*}
\]
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Then (4.3) implies that \( d \) must be given by the explicit formula
\[
d\omega = \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},
\]
which establishes uniqueness.

Proof of local existence of \( d \): We start by proving existence when \( M = U \) is
the domain of a local coordinate system \((x^1, \ldots, x^n)\). In this case, we use (4.4)
to define \( d \) anc we need only check that it satisfies the axioms. Axiom 1 is
immediate. For Axiom 3, we first check that
\[
d(fg) = gdf + fdg
\]
by the Leibniz rule for ordinary differentiation. If
\[
\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \phi = \sum_{j_1 < \cdots < j_{\ell}} g_{j_1 \cdots j_{\ell}} dx^{j_1} \wedge \cdots \wedge dx^{j_{\ell}},
\]
then
\[
\omega \wedge \phi = \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_{\ell}} f_{i_1 \cdots i_k} g_{j_1 \cdots j_{\ell}} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{\ell}}.
\]
Thus
\[
d(\omega \wedge \phi) = \sum_{i_1 < \cdots < i_k} \sum_{j_1 < \cdots < j_{\ell}} (df_{i_1 \cdots i_k} g_{j_1 \cdots j_{\ell}} + f_{i_1 \cdots i_k} dg_{j_1 \cdots j_{\ell}})
\]
\[
\cdot dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{\ell}}.
\]
We can rewrite this as
\[
d(\omega \wedge \phi) = \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{\ell}}
\]
\[
+ (-1)^k \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dg_{j_1 \cdots j_{\ell}} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{\ell}},
\]
because we have to move \( dg_{j_1 \cdots j_{\ell}} \) past \( k \) of the \( dx^i \)'s. But this in turn can be
rewritten as
\[
d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^k \omega \wedge d\phi,
\]
which is Axiom 3.

Finally, for the proof of Axiom 2 we use equality of mixed partial derivatives.
First we check that if \( f \in \Omega^1(U) \), then
\[
d(df) = d \left( \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i \right) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j
\]
\[
= \sum_{i<j} \frac{\partial^2 f}{\partial x^i \partial x^j} (dx^i \wedge dx^j + dx^j \wedge dx^i) = 0,
\]
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thereby verifying a special case of Axiom 2. We then use this special case together with Axiom 3 to verify that

\[
(d \circ d) \left( \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right)
= d \left( \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right)
= \sum_{i_1 < \cdots < i_k} \left( df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} - \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge d (dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \right) = 0,
\]

which establishes Axiom 2 in general.

Proof of global existence of $d$: We cover $M$ by local coordinate systems

\[\{(U_\alpha, (x^1_\alpha, \ldots, x^n_\alpha)) : \alpha \in A\}.\]

Local existence implies that on each $U_\alpha$ we have a collection of $\mathbb{R}$-linear operators

\[d_\alpha : \Omega^k(U_\alpha) \rightarrow \Omega^{k+1}(U_\alpha)\]

which satisfy the three axioms. Over $U_\alpha \cap U_\beta$, we have $d_\alpha = d_\beta$ by local uniqueness. Thus the locally defined $d_\alpha$’s fit together to give global operators

\[d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)\]

which satisfy the axioms. This finishes the proof of Theorem 2.3.1.

**Examples:** We suppose that $M = \mathbb{R}^3$ with Euclidean coordinates $(x, y, z)$ and with the dot product as Riemannian metric. The dot product establishes a “index lowering” isomorphism at each point, represented in musical terminology by

\[\flat : T_p^* \mathbb{R}^3 \rightarrow T_p \mathbb{R}^3 \text{ and defined by } v \in T_p^* \mathbb{R}^3 \mapsto (w \mapsto v \cdot w) \in T_p \mathbb{R}^3,\]

with an inverse which “raises the index,”

\[\sharp : T_p^* \mathbb{R}^3 \rightarrow T_p \mathbb{R}^3.\]

In addition, we have the standard basis

\[i, j, k \text{ for } T_p \mathbb{R}^3,\]

which extends to an $\mathcal{F}(\mathbb{R}^3)$ module basis of constant vector fields

\[i, j, k.\]
But for now, we will be somewhat more informal when considering
\[ dx = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}, \quad N dA = (dy \land dz)\mathbf{i} + (dz \land dx)\mathbf{j} + (dx \land dy)\mathbf{k}, \quad (4.5) \]
which should actually be sections of an appropriate vector bundle over \( \mathbb{R}^3 \). The vector bundle terminology might seem like unnecessary baggage when dealing just with Euclidean space.

If \( f \in \Omega^0(\mathbb{R}^3) \), then
\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = (\nabla f) \cdot dx, \]
where
\[ \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \]
is the gradient of \( f \), which is regarded as a contravariant vector field. We could write
\[ df = \flat(\nabla f), \]
the Riemannian metric (or dot product) being used to identify the gradient with the differential in ordinary calculus of several variables.

Suppose that we are given a contravariant vector field
\[ \mathbf{V} = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \]
on \( \mathbb{R}^3 \). We can lower the index, thus constructing the corresponding one-form
\[ \omega = \flat(\mathbf{V}) = \mathbf{V} \cdot dx = Pdx + Qdy + Rdz \in \Omega^1(\mathbb{R}^3). \]

Then
\[ d\omega = dP \land dx + dQ \land dy + dR \land dz = \]
\[ \left( \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \land dx + \left( \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy + \frac{\partial Q}{\partial z} dz \right) \land dy + \left( \frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz \right) \land dz. \]

We can use the skew-symmetry of the wedge product to simplify the last expression:
\[ d\omega = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \land dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \land dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \land dy. \]

If we use (4.5), we can rewrite this as
\[ d\omega = (\nabla \times \mathbf{V}) \cdot N dA, \]
where \( \nabla \times \mathbf{V} \) is the curl of \( \mathbf{V} \) as studied in vector calculus.
Given a contravariant vector field

\[ \mathbf{V} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}, \]

we can also construct the two-form

\[ \omega = \mathbf{V} \cdot \text{NdA} = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \in \Omega^2(\mathbb{R}^3). \]

In this case,

\[ d\omega = dP \wedge dy \wedge dz + dQ \wedge dz \wedge dx + dR \wedge dx \wedge dy = \cdots = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz. \]

We can rewrite this as

\[ d\omega = (\nabla \cdot \mathbf{V}) dx \wedge dy \wedge dz, \]

where \( \nabla \cdot \mathbf{V} \) is the divergence of \( \mathbf{V} \).

Finally, we mention that the wedge product and the exterior derivative are natural under smooth maps. A smooth map \( F : M \rightarrow N \) induces an \( \mathbb{R} \)-linear map

\[ F^* : \Omega^k(N) \rightarrow \Omega^k(N) \]

by

\[ F^*(\omega)(p)(v_1, \ldots, v_k) = \omega(F(p))(F_p(v_1), \ldots, F_p(v_k)), \]

for \( v_1, \ldots, v_k \in T_pM. \)

**Proposition 4.3.3.** If \( F : M \rightarrow N \) is a smooth map, then

\[ F^*(\omega \wedge \phi) = F^*(\omega) \wedge F^*(\phi), \quad F^*(d\omega) = d(F^*\omega). \]

We leave the proof that \( F^* \) preserves wedge products to the reader. The fact that

\[ F^*(df) = d(F^*f), \quad \text{for } f \in \Omega^0(M) \]

follows from Proposition 1.5.4. We next check that

\[ F^*(d\omega) = d(F^*\omega) \quad \text{for } f \in \Omega^k(U), \]

when \( U \) is the domain of a local coordinate system \( (x^1, \ldots, x^n) \). This allows us to use the local coordinate formula (4.4). If

\[ \omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \]

then
then
\[ d\omega = \sum_{i_1 < \cdots < i_k} df_{i_1 \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \]
so
\[ F^*(d\omega) = \sum_{i_1 < \cdots < i_k} F^*(df_{i_1 \cdots i_k}) \wedge F^*dx^{i_1} \wedge \cdots \wedge F^*dx^{i_k} = \sum_{i_1 < \cdots < i_k} d(f_{i_1 \cdots i_k} \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F). \] (4.6)

On the other hand,
\[ F^*\omega = \sum_{i_1 < \cdots < i_k} F^*f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{i_1 < \cdots < i_k} (f_{i_1 \cdots i_k} \circ F)d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F), \]
so it follows from the axioms that
\[ d(F^*\omega) = \sum_{i_1 < \cdots < i_k} d(f_{i_1 \cdots i_k} \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F). \]
Comparison with (4.6) now yields
\[ F^* \circ d = d \circ F^* \] on \( U \). Finally, since both \( d \) and \( F^* \) are local operators (Lemma 2.3.2), we see that \( F^* \circ d = d \circ F^* \) on \( M \) itself. QED

Note that if \( \omega \in \Omega^k(M) \) and \( X_1, \ldots, X_k \) are smooth vector fields on \( M \), we can define a smooth function \( \omega(X_1, \ldots, X_k) \) on \( M \) by
\[ \omega(X_1, \ldots, X_k)(p) = \omega(p)(X_1(p), \ldots, X_k(p)). \]

**Exercise VI.** (Due Wednesday, December 3.) Show that if \( X \) and \( Y \) are smooth vector fields on \( M \) and \( \theta \in \Omega^1(M) \) is a smooth one-form, then
\[ d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]). \]
Hint: Use local coordinates.

**Exercise VII.** (Due Wednesday, December 3.) a. Recall that we can define coordinates
\[ x^i_j : GL(n, \mathbb{R}) \to \mathbb{R} \text{ by } x^i_j \left( \begin{array}{cccc} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{ni} & \cdots & a_{nn} \end{array} \right) = a^i_j. \]
Similarly, we can define
\[ y^j_i : GL(n, \mathbb{R}) \to \mathbb{R} \text{ by } y^j_i(A) = x^j_i(A^{-1}). \]
If $L_A : GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is the diffeomorphism defined by $L_A(B) = AB$ and
\[
\omega_j^i = \sum_{k=1}^{n} y_k^i dx_j^k, \quad \text{show that} \quad (L_A)^* \omega_j^i = \omega_j^i \quad \text{for all} \quad A \in GL(n, \mathbb{R}).
\]
b. Show that if
\[
X_i^j = \sum_{k=1}^{n} x_i^k \frac{\partial}{\partial x_j^k}, \quad \text{then} \quad \omega_j^i(X_k^\ell) = \begin{cases} 1, & \text{if} \ (i, j) = (\ell, k), \\ 0, & \text{otherwise}. \end{cases}
\]
Use this to show that $X_i^j$ is a left invariant vector field on $GL(n, \mathbb{R})$. (This contrasts with the earlier proof we gave in §3.3 that $X_i^j$ is left invariant.)

c. Show that
\[
d\omega_j^i = -\sum_{k=1}^{n} \omega_k^i \wedge \omega_j^k.
\]
(These are known as the equations of Maurer-Cartan.)

4.4 Integration of differential forms

What gives particular importance to the differential forms (among the more general covariant tensor fields) is the fact that they serve as integrands for multiple integrals.

We already mentioned orientability—let’s review the definition. Two smooth charts $(U, (x^1, \ldots, x^n))$ and $(V, (y^1, \ldots, y^n))$ on an $n$-dimensional smooth manifold $M$ are said to be coherently oriented if
\[
\det \left( \frac{\partial y^i}{\partial x^j} \right) > 0,
\]
where defined. We say that $M$ is orientable if it possesses an atlas of coherently oriented charts. Such an atlas is called an orientation for $M$. An oriented smooth manifold is a smooth manifold together with a choice of orientation.

Suppose that $M$ is a smooth manifold with orientation defined by the atlas $\mathcal{A}$ of coherently oriented charts. Then a chart $(V, (y^1, \ldots, y^n))$ on $M$ is said to be positively oriented if
\[
\det \left( \frac{\partial y^i}{\partial x^j} \right) > 0,
\]
where defined, for every chart $(U, (x^1, \ldots, x^n))$ in the coherently oriented atlas $\mathcal{A}$. A connected orientable manifold always has exactly two orientations.

**Local integration of $n$-forms:** Suppose that $\omega$ is a smooth $n$-form with compact support in an open subset $U$ of a smooth $n$-dimensional oriented manifold $M$. If $\phi = (x^1, \ldots, x^n)$ are positively oriented coordinates on $U$, we can write
\[
f \ dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.
\]
We can then define the integral of $\omega$ over $U$ by the formula

$$\int_U \omega = \int_{\mathbb{R}^n} (f \circ \phi^{-1}) dx^1 \cdots dx^n.$$

Thus to integrate an $n$-form over an oriented $n$-manifold, we essentially just leave out the wedges and take the ordinary Riemann integral.

We need to check that this definition is independent of choice of positively oriented smooth coordinates. To do this, note that if $\psi = (y^1, \ldots, y^n)$ is a second positively oriented coordinate system on $U$, then

$$dx^i = \sum_{j=1}^n \frac{\partial x^i}{\partial y^j} dy^j,$$

and hence

$$dx^1 \wedge \cdots \wedge dx^n = \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{\partial x^1}{\partial y^{j_1}} \cdots \frac{\partial x^n}{\partial y^{j_n}} dy^{j_1} \wedge \cdots \wedge dy^{j_n}$$

$$= \sum_{\sigma \in S_n} \frac{\partial x^1}{\partial y^{\sigma(1)}} \cdots \frac{\partial x^n}{\partial y^{\sigma(n)}} dy^{\sigma(1)} \wedge \cdots \wedge dy^{\sigma(n)},$$

which yields

$$dx^1 \wedge \cdots \wedge dx^n = \sum_{\sigma \in S_n} \text{sgn}\sigma \frac{\partial x^1}{\partial y^{\sigma(1)}} \cdots \frac{\partial x^n}{\partial y^{\sigma(n)}} dy^{1} \wedge \cdots \wedge dy^{n},$$

Recall that if $A = (a^i_j)$ is an arbitrary $n \times n$ matrix,

$$\det A = \sum_{\sigma \in S_n} \text{sgn}\sigma a^1_{\sigma(1)} \cdots a^n_{\sigma(n)},$$

and since the two coordinate systems are coherently oriented,

$$dx^1 \wedge \cdots \wedge dx^n = \det \left( \frac{\partial x^i}{\partial y^j} \right) dy^1 \wedge \cdots \wedge dy^n = \left| \det \left( \frac{\partial x^i}{\partial y^j} \right) \right| dy^1 \wedge \cdots \wedge dy^n.$$

Thus the two expressions for the integral of $\omega$ over $U$ will agree if and only if

$$\int_{\mathbb{R}^n} (f \circ \phi^{-1}) dx^1 \cdots dx^n = \int_{\mathbb{R}^n} (f \circ \psi^{-1}) \left| \det \left( \frac{\partial x^i}{\partial y^j} \right) \right| dy^1 \cdots dy^n.$$

If we set $\tilde{f} = f \circ \phi^{-1}$, we can write this as

$$\int_{\mathbb{R}^n} \tilde{f} dx^1 \cdots dx^n = \int_{\mathbb{R}^n} (\tilde{f} \circ (\phi \circ \psi^{-1})) \left| \det \left( \frac{\partial y^i}{\partial x^j} \right) \right| dy^1 \cdots dy^n.$$  \hspace{1cm} (4.7)

Since

$$\phi \circ \psi^{-1}(y^1, \ldots, y^n) = (x^1, \ldots, x^n),$$

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(4.7) is just the formula for change of variable in a multiple integral, familiar from vector calculus, a proof of which can be found in any undergraduate text on real analysis, such as Rudin [16], Theorem 9.32. This change of variables formula implies that local integration is indeed well-defined, and does not depend on the choice of local coordinates.

Local integration of differential forms has the usual linearity property:

\[ \int_U (c_1 \omega_1 + c_2 \omega_2) = c_1 \int_U \omega_1 + c_2 \int_U \omega_2, \]

where \( c_1 \) and \( c_2 \) are constants and \( \omega_1 \) and \( \omega_2 \) are \( n \)-forms with compact support within \( U \).

**Global integration of \( n \)-forms:** This is defined by the “standard procedure” of piecing together with a partition of unity. Suppose that \( \omega \) is a smooth \( n \)-form with compact support on a smooth oriented \( n \)-dimensional manifold. Choose an open covering \( \{ U_\alpha : \alpha \in \mathcal{A} \} \) such that each \( U_\alpha \) is the domain of a positively oriented coordinate system, and let \( \{ \psi_\alpha : \alpha \in \mathcal{A} \} \) be a partition of unity subordinate to the open cover \( \{ U_\alpha : \alpha \in \mathcal{A} \} \). Then we can apply the preceding construction to each differential form \( \psi_\alpha \omega \) and define the integral of \( \omega \) over \( M \) by the formula

\[ \int_M \omega = \sum_{\alpha \in \mathcal{A}} \int_{U_\alpha} \psi_\alpha \omega. \]

The sum is actually finite, since \( \omega \) has compact support and the supports of \( \{ \psi_\alpha : \alpha \in \mathcal{A} \} \) are locally finite.

We need to check that this definition is independent of choice of cover and partition of unity. Suppose that \( \{ V_\beta : \beta \in \mathcal{B} \} \) is another open cover by domains of positively oriented coordinate systems and that \( \{ \eta_\beta : \beta \in \mathcal{B} \} \) is a subordinate partition of unity. Then

\[ \sum_{\alpha \in \mathcal{A}} \int_{U_\alpha} \psi_\alpha \omega = \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} \int_{U_\alpha \cap V_\beta} \psi_\alpha \eta_\beta \omega = \sum_{\beta \in \mathcal{B}} \int_{V_\beta} \eta_\beta \omega. \]

We can now make precise what we mean when we say that \( k \)-forms are integrands for integrals over \( k \)-dimensional manifolds. Suppose that \( M \) is an \( n \)-dimensional manifold and \( \Sigma \) is a compact \( k \)-dimensional oriented submanifold of \( M \) with inclusion \( \iota : S \to M \). If \( \omega \in \Omega^k(M) \), we can define the integral of \( \omega \) over \( \Sigma \),

\[ \int_{\Sigma} \iota^* \omega. \]

In fact, if \( F : \Sigma \to M \) is simply a smooth map, we can define the integral of \( \omega \) over the singular manifold \( \langle \Sigma, F \rangle \) as

\[ \int_{\Sigma} F^* \omega. \]

This notion of integration over singular manifolds includes the line integrals and surface integrals encountered in vector calculus.
4.5 Theorem of Stokes

The Theorem of Stokes generalizes Green’s Theorem from several variable calculus to manifolds of arbitrary dimension. To set up the context, we first define manifolds with boundary. Let
\[ R^n_- = \{(x^1, \ldots, x^n) : x^1 \leq 0\}, \quad \partial R^n_- = \{(x^1, \ldots, x^n) : x^1 = 0\}. \]

In this section, we will identify \( \partial R^n_- \) with \( R^{n-1} \).

**Definition.** An \( n \)-dimensional smooth manifold with boundary is a second countable Hausdorff topological space \( M \) together with a collection \( A = \{(U_\alpha, \phi_\alpha) : \alpha \in A\} \) such that:
1. Each \( \phi_\alpha \) is a homeomorphism from an open subset \( U_\alpha \) of \( M \) onto an open subset of \( \mathbb{R}^n \).
2. \( \bigcup \{U_\alpha : \alpha \in A\} = M \).
3. \( \phi_\beta \circ \phi_\alpha^{-1} \) is \( C^\infty \) where defined.

The boundary of \( M \) is
\[ \partial M = \{p \in M : \phi_\alpha(p) \in \partial R^n_- \text{ for some } \alpha \in A\}. \]

**Lemma 4.5.1.** If \( \phi_\alpha(p) \in \partial R^n_- \) for some \( \alpha \in A \), then \( \phi_\beta(p) \in \partial R^n_- \) for all \( \beta \in A \) for which \( \phi_\beta(p) \) is defined.

Proof: Suppose on the contrary that \( \phi_\beta(p) \) lies in the interior of \( \mathbb{R}^n_- \) for some \( \beta \). Note that \( \phi_\alpha \circ \phi_\beta^{-1} \) has a nonsingular differential at \( \phi_\beta(p) \). Hence by the inverse function theorem, \( \phi_\alpha \circ \phi_\beta^{-1} \) maps some neighborhood \( V_\beta \) of \( \phi_\beta(p) \) onto an open neighborhood \( V_\alpha \) of \( \phi_\alpha(p) \). Then \( V_\alpha \) is open in \( \mathbb{R}^n \) and \( V_\alpha \subseteq \mathbb{R}^n_- \). Hence \( \phi_\alpha(p) \) lies in the interior of \( \mathbb{R}^n_- \), a contradiction.

If \( \alpha \in A \), we let \( V_\alpha = U_\alpha \cap \partial M \) and let \( \psi_\alpha = \phi_\alpha|V_\alpha \). The lemma shows that \( \psi_\alpha \) is \( \mathbb{R}^{n-1} \)-valued. Thus \( \partial M \) become a \((n-1)\)-dimensional smooth manifold with smooth atlas \( \{V_\alpha, \psi_\alpha : \alpha \in A\} \).

**Lemma 4.5.2. (Collar Neighborhood Lemma)** If \( M \) is a smooth manifold with boundary \( \partial M \), then there is an open neighborhood of \( \partial M \) which is diffeomorphic to \( \partial M \times [0, \varepsilon) \).

Although we don’t give a complete proof, we can sketch the simple idea: If \( (U_\alpha, (x^1_\alpha, \ldots, x^n_\alpha)) \) is an element of \( \mathcal{A} \), we can let
\[ X_\alpha = -\frac{\partial}{\partial x^n_\alpha} \text{ on } U_\alpha. \]

Using a partition of unity \( \{\psi_\alpha : \alpha \in A\} \) subordinate to the atlas \( \mathcal{A} \), we can then construct the smooth vector field
\[ X = \sum_{\alpha \in A} \psi_\alpha X_\alpha, \]
and check that this vector field is nonzero and points into $M$ at points of $\partial M$.
Let $\{\phi_t : t \in \mathbb{R} \}$ be the one-parameter group of diffeomorphisms for $X$, which is globally defined because $M$ is compact. For some $\varepsilon > 0$, we can then define

$$
\Phi : \partial M \times [0, \varepsilon) \longrightarrow M \quad \text{by} \quad \Phi(p, t) = \phi_t(p)
$$

and check that it is diffeomorphism for $\varepsilon > 0$ sufficiently small. The reader could refer to Milnor [11], Theorem 3.5 for more details.

**Orientation:** Suppose that $M$ is an oriented $n$-dimensional manifold with boundary, so that $M$ has an atlas $\mathcal{A}$ whose elements are coherently oriented. Thus if $(U, (x^1, \ldots, x^n))$ and $(V, (y^1, \ldots, y^n))$ are two elements of $\mathcal{A}$, then

$$
\det \left( \frac{\partial y^i}{\partial x^j} \right) > 0,
$$

where defined. Then $(U \cap \partial M, (x^2, \ldots, x^n))$ and $(V \cap \partial M, (y^2, \ldots, y^n))$ are smooth coordinate systems on $\partial M$. We claim that they are also coherently oriented. Indeed, if $p \in \partial M \cap (U \cap V)$,

$$
\frac{\partial y^1}{\partial x^i}(p) = 0, \quad \text{for } 2 \leq i \leq n, \quad \text{since } \frac{\partial}{\partial x^1} \bigg|_p
$$

is tangent to $\partial M$ and $y^1$ is constant along $\partial M$. On the other hand, $(\partial/\partial x^1)|_p$ points out of $M$, that is in the direction of increasing $y^1$. Hence

$$
\det \begin{pmatrix}
(\partial y^1/\partial x^1)(p) & 0 & \cdots & 0 \\
* & (\partial y^2/\partial x^2)(p) & \cdots & (\partial y^2/\partial x^n)(p) \\
* & \cdots & \cdots & \cdots \\
* & (\partial y^n/\partial x^2)(p) & \cdots & (\partial y^n/\partial x^n)(p)
\end{pmatrix} > 0
$$

implies that

$$
\det \begin{pmatrix}
(\partial y^2/\partial x^2)(p) & \cdots & (\partial y^2/\partial x^n)(p) \\
\cdots & \cdots & \cdots \\
(\partial y^n/\partial x^2)(p) & \cdots & (\partial y^n/\partial x^n)(p)
\end{pmatrix} > 0,
$$

and hence $(U \cap \partial M, (x^2, \ldots, x^n))$ and $(V \cap \partial M, (y^2, \ldots, y^n))$ are indeed coherently oriented, just as we claimed.

In other words, an orientation on a smooth manifold with boundary induces an orientation on its boundary, called the **induced orientation**.

**Theorem 4.5.3. (Stokes)** Let $M$ be an oriented smooth manifold with boundary $\partial M$, and suppose that $\partial M$ is given the induced orientation. Let $\iota : \partial M \rightarrow M$ be the inclusion map. If $\theta$ is a smooth $(n-1)$-form on $M$ with compact support, then

$$
\int_{\partial M} \iota^* \theta = \int_M d\theta. \quad (4.8)
$$
Proof: Cover $M$ by positively oriented charts $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ such that if $\alpha \in A$, either

$$\phi_\alpha(U_\alpha) = (a_1^\alpha, b_1^\alpha) \times \cdots \times (a_n^\alpha, b_n^\alpha), \quad \text{or} \quad (4.9)$$

$$\phi_\alpha(U_\alpha) = (a_1^\alpha, 0] \times \cdots \times (a_n^\alpha, b_n^\alpha). \quad (4.10)$$

Let $\{\psi_\alpha : \alpha \in A\}$ be a partition of unity subordinate to the open cover $\{U_\alpha : \alpha \in A\}$.

It will suffice to prove Stokes’ Theorem for the special case where the support of $\theta$ is contained in some $U_\alpha$ for $\alpha \in A$. Indeed, assuming this special case, we find that if $\theta$ is an arbitrary $(n - 1)$-form with compact support,

$$\int_{\partial M} \iota^* \theta = \int_{\partial M} \iota^* \sum_{\alpha \in A} \psi_\alpha \theta = \sum_{\alpha \in A} \int_{\partial M} \iota^*(\psi_\alpha \theta) = \sum_{\alpha \in A} \int_M d(\psi_\alpha \theta) = \int_M d \left( \sum_{\alpha \in A} \psi_\alpha \theta \right) = \int_M d\theta.$$

Thus it suffices to prove Stokes’ Theorem in the special case where the support of $\theta$ is contained in $U$, where $U$ is the domain of a chart $(U, \phi)$ of type (4.9) or (4.10). Since the case of type (4.9) is simpler, we consider only the case of type (4.10), and suppose that

$$\phi(U) = (a_1, 0] \times \cdots \times (a_n, b^n).$$

Indeed, we can assume that $\phi$ is the identity and that $U$ itself is a rectangular set in $\mathbb{R}^n$.

Thus we let $(x^1, \ldots, x^n)$ be the usual rectangular cartesian coordinates on $U$ and that the inclusion $\iota : \partial \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\iota(x^2, \ldots, x^n) = (0, x^2, \ldots, x^n).$$

If $\theta$ is a smooth $(n - 1)$-form on $U$ with compact support in $U$, we can write

$$\theta = \sum_{i=1}^n (-1)^{i-1} f_i d\omega^i_{i-1} \wedge \cdots \wedge dx^i_{i+1} \wedge \cdots \wedge dx^n.$$

Then since $x^1$ is constant on $\partial U$, $\iota^*(dx^1) = 0$, and hence

$$\iota^* \theta = (f_1 \circ \iota)dx^2 \wedge \cdots \wedge dx^n,$$

while

$$d\theta = \sum_{i=1}^n \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.$$
To verify Stokes’ Theorem, we need to calculate the two integrals appearing in (4.8) with $M = U$. For the right-hand integral, we obtain
\[
\int_U d\theta = \sum_{i=1}^{n} \int_U \frac{\partial f_i}{\partial x^i}(x^1, \ldots, x^n) dx^1 \cdots dx^n
\]
\[
= \sum_{i=1}^{n} \int_{a^i}^{b^i} \int_a^b \frac{\partial f_i}{\partial x^i}(x^1, \ldots, x^n) dx^1 \cdots dx^n.
\]
Now we note that for $2 \leq i \leq n$,
\[
\int_{a^i}^{b^i} \int_a^b \frac{\partial f_i}{\partial x^i}(x^1, \ldots, x^n) dx^1 \cdots dx^n
\]
\[
= \int_{a^i}^{b^i} \int_{a^{i-1}}^{b^{i-1}} \int_{a}^{b} \left[ f(x^1, \ldots, b^i, \ldots, x^n) - f(x^1, \ldots, a^i, \ldots, x^n) \right] dx^1 \cdots dx^{i-1} dx^{i+1} \cdots dx^n = 0,
\]
because $f_i$ has compact support in $U$, while in the remaining case, we get
\[
\int_{a^1}^{b^1} \int_a^{b} \frac{\partial f_1}{\partial x^1}(x^1, \ldots, x^n) dx^1 \cdots dx^n
\]
\[
= \int_{a^1}^{b^1} \int_{a^2}^{b^2} \left[ f(0, \ldots, x^n) - f(a^1, \ldots, x^n) \right] dx^2 \cdots dx^n
\]
\[
= \int_{a^1}^{b^1} \int_a^{b^2} f(0, \ldots, x^n) dx^2 \cdots dx^n.
\]
Thus
\[
\int_U d\theta = \int_{a^1}^{b^1} \int_a^{b^2} f(0, \ldots, x^n) dx^2 \cdots dx^n. \tag{4.11}
\]
On the other hand,
\[
\int_{\partial U} i^* \theta = \int_{\partial U} (f_1 \circ i)(x^2, \ldots, x^n) dx^2 \cdots dx^n
\]
\[
= \int_{a^2}^{b^2} \int_{a^n}^{b^n} f(0, \ldots, x^n) dx^2 \cdots dx^n. \tag{4.12}
\]
Stokes’ Theorem now follows from (4.11) and (4.12).

**Examples.** Suppose that
\[
M = D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\},
\]
a smooth manifold which has boundary
\[
\partial M = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.
\]
Note that if we give the usual orientation to \( \mathbb{R}^2 \) which decrees that \((x, y)\) is a positively oriented coordinate system, then induced orientation on the boundary requires that it be traversed in the counterclockwise direction. If \( \theta = Pdx + Qdy \) for functions \( P \) and \( Q \), then

\[
d\theta = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy
\]

and Theorem 2.5.3 specializes to

\[
\int_{S^1} \iota^* \theta = \int_{S^1} \iota^* (Pdx + Qdy) = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy
= \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy,
\]

which is familiar from vector calculus as Green’s Theorem. Other special cases of Theorem 4.5.3 give the Divergence Theorem and the classical Stokes Theorem from vector calculus.

Indeed, suppose that \( \Sigma \) is a compact subset of the \((u, v)\)-plane \( \mathbb{R}^2 \) which is made up of a finite collection of smooth Jordan curves which comprise \( \partial \Sigma \). Then \( \Sigma \) is an oriented compact two-dimensional smooth manifold with boundary \( \partial \Sigma \).

Suppose next that \( F : \Sigma \to \mathbb{R}^3 \) is a smooth map, and

\[
\omega = V \cdot dx = Pdx + Qdy + Rdz \in \Omega^1(\mathbb{R}^3).
\]

Then

\[
d\omega = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy
= (\nabla \times V) \cdot N dA,
\]

where \( N dA = (dy \wedge dz)\mathbf{i} + (dz \wedge dx)\mathbf{j} + (dx \wedge dy)\mathbf{k} \).

We now apply the Theorem of Stokes and Proposition 4.3.3 to conclude that

\[
\int_{\partial \Sigma} V \cdot dx = \int_{\partial \Sigma} Pdx + Qdy + Rdz
= \int_{\partial \Sigma} \iota^* F^* (\omega) = \int_{\Sigma} dF^* \omega = \int_{\Sigma} F^* \omega = \int_{\Sigma} F^* ((\nabla \times V) \cdot N dA).
\]

With some effort, one can show that the integral on the right is the usual flux integral of \( \nabla \times V \) as studied in vector calculus, so we recover the usual Stokes’ Theorem in \( \mathbb{R}^3 \).

**4.6 De Rham cohomology**

The de Rham cohomology is perhaps the easiest introduction to the ideas of algebraic topology, and has the advantage of being well-adapted to applications of topology to differential geometry and physics.
The basic idea of algebraic topology is expressed within the language of categories and functors, terms we will use rather loosely. The goal of algebraic topology is to construct functors from the category of smooth manifolds and smooth maps, or more generally the category of topological spaces and continuous maps, to some algebraic category, such as the category of abelian groups and group homomorphisms, the category of finite-dimensional vector spaces and linear maps, or the category of \( \mathbb{R} \)-algebras and \( \mathbb{R} \)-algebra homomorphisms. These functors often translate difficult topological problems into algebraic problems which are easier to solve.

We give a brief introduction to de Rham theory, and refer to Bott and Tu [3] for a more detailed treatment. We remark that de Rham cohomology is closely related to the singular homology and cohomology as studied by topologists, and treated in Chapters 2 and 3 of the book by Hatcher [6].

Suppose that \( M \) is a smooth manifold, possibly with boundary. We say that an element \( \omega \in \Omega^k(M) \) is closed if \( d\omega = 0 \) and exact if \( \omega = d\theta \) for some \( \theta \in \Omega^{k-1}(M) \), and let

\[
Z^k(M) = \{ \text{closed elements of } \Omega^k(M) \} = \{ \omega \in \Omega^k(M) : d\omega = 0 \},
\]

\[
B^k(M) = \{ \text{exact elements of } \Omega^k(M) \} = \{ \omega \in \Omega^k(M) : \omega \in d(\Omega^{k-1}(M)) \}.
\]

Since \( d \circ d = 0 \), \( B^k(M) \subseteq Z^k(M) \) and we can form the quotient space, a vector space over \( \mathbb{R} \).

**Definition.** The *de Rham cohomology* of \( M \) of degree \( k \) is the quotient vector space

\[
H^k_{dR}(M; \mathbb{R}) = \frac{Z^k(M)}{B^k(M)}.
\]

If \( \omega \in Z^k(M) \), we let \([\omega]\) denote its *cohomology class*, the equivalence class of \( \omega \) in the quotient \( H^k_{dR}(M; \mathbb{R}) \). We say that

\[
\beta_k(M) = \dim H^k_{dR}(M; \mathbb{R})
\]

is the \( k \)-th *Betti number* of \( M \).

Often \( H^k_{dR}(M; \mathbb{R}) \) is finite-dimensional, so \( \beta_k \) is a finite nonnegative integer. Roughly speaking, we think of \( \beta_k \) as the number of \( k \)-dimensional holes in \( M \).

**Example 1.** If \( M \) is a smooth manifold with finitely many connected components, then \( Z^0(M) \) is just the space of functions which are constant on each component, while \( B^0(M) = 0 \), so

\[
H^0_{dR}(M; \mathbb{R}) \cong \bigoplus_{i=1}^k \mathbb{R},
\]

where \( k \) is the number of components of \( M \).
Example 2. The reader may recall the method of exact differentials for solving differential equations. This method states that if
\[ \omega = M \, dx + N \, dy \in \Omega^1(\mathbb{R}^2) \]
satisfies the integrability condition \( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 \),
then \( \omega = df \) for some smooth real-valued function \( f \) on \( \mathbb{R}^2 \). In that case,
\[ f = c \]
is a solution to the DE \( M \, dx + N \, dy = 0 \).
The integrability condition is simply the statement that \( d\omega = 0 \) so \( \omega \) is closed,
and the fact that this implies that \( \omega \) is exact is simply the assertion that \( H^1_{dR}(\mathbb{R}^2; \mathbb{R}) = 0 \), a special case of the Poincaré Lemma to be presented in the next section.
Suppose on the other hand, that \( M = \mathbb{R}^2 - \{(0, 0)\} \) and
\[ \omega = \frac{x \, dy - y \, dx}{x^2 + y^2} \in \Omega^1(M). \]
Note that the denominator in this expression is nonzero at all points of \( M \),
since \( M \) excludes the point \((0, 0, 0)\) where the expression defining \( \omega \) blows up.
A straightforward calculation shows that \( \omega \) is closed, but if we let \( S^1 \) be the unit circle \( x^2 + y^2 = 1 \) with counterclockwise parametrization, we find using techniques of vector calculus that
\[ \int_{S^1} \iota^* \omega = 2\pi. \]
where \( \iota : S^1 \subseteq \mathbb{R}^2 \) is the inclusion. Let us recall how the calculations go: if we define a parametrization \( \iota : [0, 2\pi] \rightarrow \mathbb{R}^2 \) of \( S^1 \subseteq \mathbb{R}^2 \) by
\[ (x \circ \iota)(t) = \cos t, \quad (y \circ \iota)(t) = \sin t. \]
Then
\[ \iota^* dx = -\sin t \, dt, \quad \iota^* dy = \cos t \, dt, \quad \iota^* \omega = \iota^* \left( \frac{x \, dy - y \, dx}{x^2 + y^2} \right) = dt. \]
Thus
\[ \int_{S^1} \iota^* \omega = \int_0^{2\pi} dt = 2\pi \]
as claimed. (Of course, \( t \) is really only a well-behaved coordinate on \( S^1 \) minus a point, but since \( \omega \) is smooth and bounded, it suffices for calculating the integral.)
It follows from Stokes’ Theorem that \( \omega \) cannot be exact, and \( H^1_{dR}(M; \mathbb{R}) \) is nonzero. The upshot is that the one-dimensional de Rham cohomology detects the hole that is missing at the origin.

Example 3. Recall that any smooth manifold \( M \) has an imbedding \( F : M \rightarrow \mathbb{R}^N \) for some \( N \) by the Whitney imbedding theorem, and we can use the Euclidean metric on \( \mathbb{R}^N \) to construct a Riemannian metric on \( M \). Let \( (M, \langle \cdot, \cdot \rangle) \)
be an $n$-dimensional oriented Riemannian manifold (constructed in this or some other way), $(x^1, \ldots, x^n)$ a positively oriented coordinate system defined on an open subset $U$ of $M$. If

$$\langle \cdot, \cdot \rangle = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j \quad \text{and} \quad g = \det(g_{ij}),$$

we use the Riemannian metric to define the volume form on $U$

$$\Omega_U = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n.$$

If $(\tilde{x}^1, \ldots, \tilde{x}^n)$ a second positively oriented coordinate system defined on an open subset $\tilde{U}$ of $M$,

$$\langle \cdot, \cdot \rangle = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j = \sum_{k,\ell=1}^{n} \tilde{g}_{k\ell} d\tilde{x}^k \otimes d\tilde{x}^\ell \quad \text{on} \quad U \cap \tilde{U}.$$

It follows that

$$g_{ij} = \sum_{k,\ell=1}^{n} \tilde{g}_{k\ell} \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^\ell}{\partial x^j},$$

so

$$\det(g_{ij}) = \det(\tilde{g}_{k\ell}) \left[ \det \left( \frac{\partial \tilde{x}^k}{\partial x^i} \right) \right]^2 \quad \text{or} \quad g = \tilde{g} \det \left( \frac{\partial \tilde{x}^k}{\partial x^i} \right),$$

and the volume forms for two different positively oriented coordinate systems agree on overlaps,

$$\Omega_U = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n = \sqrt{\tilde{g}} d\tilde{x}^1 \wedge \cdots d\tilde{x}^n = \Omega_{\tilde{U}} \quad \text{on} \quad U \cap \tilde{U}.$$

Thus the locally defined volume forms fit together to yield a global volume form $\Omega_M \in \Omega^n(M)$. If $M$ is compact and has empty boundary, we could define its Riemannian geometry volume by the formula

$$\text{Volume of } M = \int_M \Omega_M.$$

The volume form $\Omega_M$ has degree $n$, so it must be closed, while its integral is nonzero, so it cannot be exact by Stokes’s Theorem. Hence if $M$ is a compact oriented manifold, $H^n_{dR}(M; \mathbb{R}) \neq 0$.

We have seen that if $M$ is a smooth manifold, the exterior derivative yields a sequence of vector spaces and linear maps

$$\cdots \to \Omega^k(M) \to \Omega^{k+1}(M) \to \Omega^{k+2}(M) \to \cdots,$$

the linear maps satisfying the identity $d \circ d = 0$. This is called the de Rham cochain complex, and is denoted by $\Omega^*(M)$. A smooth map $F: M \to N$ induces
a commutative ladder
\[
\begin{array}{cccccc}
\rightarrow & \Omega^k(N) & \rightarrow & \Omega^{k+1}(N) & \rightarrow & \Omega^{k+2}(N) \\
F^* & \downarrow & F^* & \downarrow & F^* & \downarrow \\
\rightarrow & \Omega^k(M) & \rightarrow & \Omega^{k+1}(M) & \rightarrow & \Omega^{k+2}(M)
\end{array}
\] (4.13)
which can be regarded as a homomorphism of cochain complexes, and denoted by \( F^*: \Omega^* (N) \to \Omega^* (M) \). It follows from the commutativity of (4.13) that the smooth map \( F \) induces a vector space homomorphism
\[
F^*: H^k_{dR} (N; \mathbb{R}) \to H^k_{dR} (M; \mathbb{R}), \quad \text{for each } k.
\]

The direct sum
\[
H^*_dR (M; \mathbb{R}) = \sum_{k=0}^{\infty} H^k_dR (M; \mathbb{R})
\]
is not only a graded vector space, but can be made into what is called a graded anticommutative algebra over \( \mathbb{R} \), the product being the so-called cup product, which is defined as follows: If
\[
x \in H^k_{dR} (M; \mathbb{R}) \quad \text{and} \quad y \in H^l_{dR} (M; \mathbb{R}),
\]
we can write \( x = [\omega] \) and \( y = [\phi] \) where \( \omega \) and \( \phi \) are closed forms. It follows from the formula
\[
d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^k \omega \wedge d\phi \quad (4.14)
\]
that the product of closed forms is closed, so \( \omega \wedge \phi \) represents an element of \( H^{k+l}_{dR} (M; \mathbb{R}) \), and we set
\[
x \cup y = [\omega] \cup [\phi] = [\omega \wedge \phi] \in H^{k+l}_{dR} (M; \mathbb{R}).
\]
Of course, we need to check that the cup product is independent of the choice of representatives. We leave this as an exercise for the enterprising reader, with the hint that one uses the following consequence of (4.14):
\[
B^k(M) \cup Z^l(M) \subseteq B^{k+l}(M), \quad Z^k(M) \cup B^l(M) \subseteq B^{k+l}(M).
\]
Finally, if \( F: M \to N \) is a smooth map, the fact that
\[
F^* (\omega \wedge \phi) = F^* (\omega) \wedge F^* (\phi) \in H^{m+l}_{dR} (M; \mathbb{R})
\]
implies that the linear map on cohomology
\[
F^*: H^*_dR (N; \mathbb{R}) \to H^*_dR (M; \mathbb{R}) \quad \text{respects the cup product:}
\]
\[
F^* ([\omega] \cup [\phi]) = F^* [\omega] \cup F^* [\phi].
\]

The correspondences
\[
M \mapsto H^*_dR (M; \mathbb{R}), \quad (F: M \to N) \mapsto (F^*: H^*_dR (N; \mathbb{R}) \to H^*_dR (M; \mathbb{R})),
\]
with \( F^* \) going the opposite direction to \( F \), satisfy the following two axioms:
1. The identity map on a given manifold $M$ induces the identity on its de Rham cohomology.

2. If $F : M \to N$ and $G : N \to P$ are smooth maps, then $(G \circ F)^* = F^* \circ G^*$.

These two identities characterize what is called a contravariant functor from the category of smooth manifolds and smooth maps to the category of graded anti-commutative $\mathbb{R}$-algebras and graded $\mathbb{R}$-algebra homomorphisms, contravariant because $F^*$ goes the opposite direction to $F$.

**Exercise VIII.** (Due Friday, December 12.) Suppose that $M = \mathbb{R}^3 - \{0\}$ with the standard coordinates $(x, y, z)$, and define $\omega \in \Omega^2(M)$ by

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Show that $\omega$ is closed but not exact. Explain how this shows that $H^2(M; \mathbb{R}) \neq 0$. Hint: Use spherical coordinates in $\mathbb{R}^3$ and integrate over the unit two-sphere just like you would in vector calculus.

**Exercise VIIIA.** (Do not hand in.) Suppose that $X$ is a smooth vector field on $M$. Define the interior product $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$\iota_X(\omega)(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}).$$

Show that if $\omega \in \Omega^k(M)$ and $\theta \in \Omega^l(M)$, then

$$\iota_X(\omega \wedge \theta) = (\iota_X \omega) \wedge \theta + (-1)^k \omega \wedge (\iota_X \theta).$$

## 4.7 The Poincaré Lemma

In order for de Rham cohomology to be useful in solving topological problems, we need to be able to compute it in important cases. There are two main steps to developing a method for calculating de Rham cohomology. The first consists of establishing the so-called Poincaré Lemma, which calculates the de Rham cohomology of a convex open subset of $\mathbb{R}^n$. The second is the Mayer-Vietoris sequence which provides a technique for computing the homology of $M$ by means of a cover $\{U_\alpha : \alpha \in A\}$ of $M$ by open sets each diffeomorphic to a convex open subset of $\mathbb{R}^n$.

**Poincaré Lemma.** If $U$ is a convex open subset of $\mathbb{R}^n$, then the de Rham cohomology of $U$ is “trivial”:

$$H^k_{dR}(U; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

This is a powerful generalization of the method of exact differentials from calculus.
Instead of proving the Poincaré Lemma directly, it is convenient to regard it as a corollary of a more powerful fact about de Rham cohomology: de Rham cohomology is invariant under smooth homotopy.

We say that smooth maps \( F, G : M \to N \) are \emph{smoothly homotopic} if there is a smooth map \( H : [0, 1] \times M \to N \) such that

\[
H(0, p) = F(p) \quad \text{and} \quad H(1, p) = G(p).
\]

**Homotopy Theorem.** Smoothly homotopic maps \( F, G : M \to N \) between smooth manifolds without boundary induce the same map on cohomology,

\[
F^* = G^* : H^k_{dR}(N; \mathbb{R}) \to H^k_{dR}(M; \mathbb{R}).
\]

To see how the Homotopy Theorem implies the Poincaré Lemma, we first note that if \( \{p_0\} \) is a single point, regarded as a zero-dimensional manifold, then

\[
\Omega^k(\{p_0\}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}
\]

and hence \( d \equiv 0 \) and

\[
H^k_{dR}(\{p_0\}; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}
\]

Next observe that if \( p_0 \) lies within \( U \), then we have an inclusion map \( \iota : \{p_0\} \to U \) and a map \( r : U \to \{p_0\} \) such that \( r \circ \iota = \text{id}_{\{p_0\}} \). On the other hand, \( \iota \circ r \) is homotopic to the identity on \( U \), because the map \( H : [0, 1] \times U \to U \) defined by

\[
H(t, p) = tp_0 + (1 - t)p \quad \text{satisfies} \quad H(0, p) = p, \quad H(1, p) = p_0.
\]

Thus functoriality implies that

\[
\iota^* : H^*_{dR}(U; \mathbb{R}) \to H^*_{dR}(\{p_0\}; \mathbb{R}) \quad \text{and} \quad r^* : H^*_{dR}(\{p_0\}; \mathbb{R}) \to H^*_{dR}(U; \mathbb{R})
\]

are both isomorphisms.

**Proof of Homotopy Theorem:** The Homotopy Theorem follows from the special case for the homotopic inclusion maps

\[
\iota_0, \iota_1 : M \to [0, 1] \times M, \quad \iota_0(p) = (0, p), \quad \iota_1(p) = (1, p).
\]

Indeed, if \( H : [0, 1] \times M \to N \) is a smooth homotopy from \( F \) to \( G \), then by definition of homotopy, \( F = H \circ \iota_0 \) and \( G = H \circ \iota_1 \), so

\[
\iota_0^* = \iota_1^* \Rightarrow F^* = \iota_0^* \circ H^* = \iota_1^* \circ H^* = G^*,
\]

proving the Homotopy Theorem.
This special case, however, can be established by integration over the fiber of the projection on the second factor \([0, 1] \times M \to M\):

**Fiber Integration Lemma.** For each nonnegative integer \(k\), there is a linear map
\[
\pi_* : \Omega^k([0, 1] \times M) \to \Omega^{k-1}(M)
\]
such that
\[
\iota^*_1 \omega - \iota^*_0 \omega = d(\pi_*(\omega)) + \pi_*(d\omega).
\] (4.15)

Proof that the Fiber Integration Lemma implies the Homotopy Lemma: It follows from (4.15) that if \(d\omega = 0\),
\[
\iota^*_1 \omega - \iota^*_0 \omega = d(\pi_*(\omega)) \Rightarrow [\iota^*_1 \omega] = [\iota^*_0 \omega],
\]
and hence on the cohomology level \(\iota^*_0 = \iota^*_1\).

Proof of the Fiber Integration Lemma. Let \(t\) be the standard coordinate on \([0, 1]\), \(T\) the vector field tangent to the \([0, 1]\) factor in \([0, 1] \times M\) such that \(dt(T) = 1\). We then define integration over the fiber
\[
\pi_* : \Omega^k([0, 1] \times M) \to \Omega^{k-1}(M) \quad \text{by} \quad \pi_*(\omega)(p) = \int_0^1 (\iota_T \omega)(t, p) dt,
\]
where the interior product \((\iota_T \omega)(t, p)\) is the element of \(\Lambda^k T^*_{(t, p)}([0, 1] \times M)\) given by the formula
\[
(\iota_T \omega)(t, p)(v_1, \ldots, v_{k-1}) = \omega(t, p)(T(t, p), v_1, \ldots, v_{k-1}),
\]
for \(v_1, \ldots, v_{k-1} \in T_{(t, p)}([0, 1] \times M)\).

The integration is possible because the exterior power at \((t, p)\) is canonically isomorphic to \(\Lambda^k T^*_{(0, p)}([0, 1] \times M)\).

Thus to finish the proof of the Fiber Exterior Integration Lemma, it remains only to establish (4.15). Let \(\{U_\alpha, \phi_\alpha\} : \alpha \in A\) be an atlas for \(M\), and let \(\{\psi_\alpha : \alpha \in A\}\) be a partition of unity subordinate to \(\{U_\alpha : \alpha \in A\}\). Since (4.15) is linear as a function of \(\omega\) it suffices to establish (4.15) for the differential forms \(\psi_\alpha \omega\) which have compact support within \(U_\alpha\). Thus let \((U, (x^1, \ldots, x^n))\) be one of the elements of the atlas, take \((t, x^1, \ldots, x^n)\) as the corresponding smooth coordinate system on \([0, 1] \times U\), and suppose that the \(k\)-form \(\omega\) has compact support within \([0, 1] \times U\). Then the above formula for \(\pi_*\) gives
\[
\omega = f(t, x) dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \Rightarrow \quad \pi_*(\omega) = 0, \quad (4.16)
\]
\[
\omega = f(t, x) dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}
\]
\[
\Rightarrow \quad \pi_*(\omega) = \left[ \int_0^1 f(u, x) du \right] dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}. \quad (4.17)
\]
Since any $k$-form with support in $[0, 1] \times U$ is a superposition of differential forms treated by (4.16) and (4.17), we need only verify the “cochain homotopy formula” (4.15) in each of these two cases.

In the first case (4.16), $\pi_* (\omega) = 0$ so $d(\pi_* (\omega)) = 0$. On the other hand,

$$d\omega = \frac{\partial f}{\partial t} \, dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \text{ (terms not involving } dt),$$

so

$$\pi_* (d\omega) = \left[ \int_0^1 \frac{\partial f}{\partial u} (u, x) du \right] dx^{i_1} \wedge \cdots \wedge dx^{i_k} = f(1, x) dx^{i_1} \wedge \cdots \wedge dx^{i_k} - f(0, x) dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \iota^*_1 (\omega) - \iota^*_0 (\omega),$$

so the formula is established in this case.

In the other case (4.17),

$$\iota^*_1 (dt) = \iota^*_0 (dt) = 0 \Rightarrow \iota^*_1 (\omega) = \iota^*_0 (\omega) = 0. \quad (4.18)$$

On the one hand,

$$d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x^j} dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$$

$$= - \sum_{j=1}^n \frac{\partial f}{\partial x^j} dt \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$$

and hence

$$\pi_* (d\omega) = - \sum_{j=1}^n \left[ \int_0^1 \frac{\partial f}{\partial x^j} (u, x) du \right] dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}. \quad (4.19)$$

On the other hand,

$$\pi_* (\omega) = \left[ \int_0^1 f(u, x) du \right] dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}},$$

so

$$d\pi_* (\omega) = \sum_{j=1}^n \frac{\partial f}{\partial x^j} \left[ \int_0^1 f(u, x) du \right] dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}$$

$$= \sum_{j=1}^n \left[ \int_0^1 \frac{\partial f}{\partial x^j} (u, x) du \right] dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}. \quad (4.20)$$

In view of (4.18), the desired identity now follows by adding (4.19) and (4.20).

**Definition.** We say that two smooth manifolds $M$ and $N$ are *smoothly homotopic equivalent* if there exist smooth maps $F : M \to N$ and $G : N \to M$ such that $G \circ F$ and $F \circ G$ are both homotopic to the identity.
For example, the cylinder is smoothly homotopy equivalent to a circle. Similarly, the punctured plane is smoothly homotopy equivalent to a circle. It follows from the Homotopy Theorem that if two manifolds are smoothly homotopy equivalent, they have the same de Rham cohomology.

4.8 Applications of de Rham theory

In this section, we give two applications of de Rham’s cohomology, first an easy proof of the Brouwer fixed point theorem, which states that a continuous map from the unit disk to itself must contain a fixed point, and second a proof that it is not possible to comb the hair on a billiard ball.

For the Brouwer fixed point theorem, we consider the unit disk

\[ D^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : (x^1)^2 + \cdots + (x^n)^2 \leq 1\}, \]

a smooth manifold with boundary, the boundary being

\[ S^{n-1} = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : (x^1)^2 + \cdots + (x^n)^2 = 1\}. \]

Lemma 4.8.1. If \( n \geq 2 \), there does not exist a smooth map \( F : D^n \to S^{n-1} \) which leaves \( S^{n-1} \) pointwise fixed.

Proof: Suppose that there were such a map \( F \), and let \( p_0 = F(0) \). Define

\[ H : [0, 1] \times S^{n-1} \to S^{n-1} \text{ by } H(t, p) = F(tp). \]

Then \( H \) is a smooth homotopy from \( c \) to \( id \) where \( c \) is the constant map which takes \( S^{n-1} \) to \( p_0 \) and \( id \) is the identity map of \( S^{n-1} \). By the Homotopy Theorem,

\[ c^* = id^* : H^{n-1}_{dR}(S^{n-1}; \mathbb{R}) \to H^{n-1}_{dR}(S^{n-1}; \mathbb{R}). \]

If \( \omega \) is a smooth \((n - 1)\)-form on \( S^{n-1} \), \( c^*(\omega) = 0 \) since \( c \) is constant, and hence \( c^* \) is the zero map on \( H^{n-1}_{dR}(S^{n-1}; \mathbb{R}) \). On the other hand, it follows from Stokes’s Theorem that \( H^{n-1}_{dR}(S^{n-1}; \mathbb{R}) \neq 0 \), so \( id^* = id \) is not the zero map, a contradiction.

Proposition 4.8.2. A smooth map \( G : D^n \to D^n \) must have a fixed point.

Proof: If \( G : D^n \to D^n \) is a smooth map with no fixed point, define \( F : D^n \to S^{n-1} \) as follows: For \( p \in D^n \), let \( L(p) \) denote the line through \( p \) and \( G(p) \) and let \( F(p) \) be the point on \( S^{n-1} \cap L(p) \) closer to \( p \) than \( G(p) \). Since \( G \) is smooth, \( F \) is also smooth. Moreover, \( F \) leaves \( S^{n-1} \) pointwise fixed, contradicting the previous lemma.

Theorem 4.8.3 (Brouwer Fixed Point Theorem). A continuous map \( f : D^n \to D^n \) must have a fixed point.
Proof: Suppose that \( f : D^n \to D^n \) is a continuous map and \( \epsilon > 0 \) is given. By the Weierstrass approximation theorem, there is a smooth polynomial map \( P : D^n \to \mathbb{R}^n \) such that \(|P(p) - f(p)| < \epsilon\) for \( p \in D^n \). Let
\[
G = \frac{1}{1 + \epsilon} P.
\]
Then \( G : D^n \to D^n \) is a smooth map which approximates \( f \) arbitrarily closely.

Suppose now that \( f : D^n \to D^n \) is a continuous map without fixed points and let
\[
\mu = \inf \{|f(p) - p| : p \in D^n\}.
\]
By the argument in the preceding paragraph, we can choose a smooth map \( G : D^n \to D^n \) such that \(|f - G| < \mu\). Then \( G \) is a smooth map without fixed points, contradicting the preceding proposition.

**Remark.** The proof of Theorem 4.8.3 illustrates that it is sometimes possible to obtain topological results involving continuous maps by approximation from the theory of smooth manifolds and smooth maps.

We next prove that there is no smooth nowhere zero vector field tangent to \( S^2 \). This makes use of the cohomology of \( SO(3) \) which we will show how to compute in the next section. The result is that
\[
H^k_{dR}(SO(3); \mathbb{R}) = H^k_{dR}(\mathbb{R}P^3; \mathbb{R}) \cong \begin{cases} \mathbb{R}, & \text{if } k = 0 \text{ or } k = 3, \\ 0, & \text{otherwise}. \end{cases} \tag{4.21}
\]

**Theorem 4.8.4.** Any smooth vector field \( X : S^2 \to TS^2 \) must have a zero.

**Proof:** We suppose that \( S^2 \) has a nowhere vanishing tangent vector field \( X : S^2 \to TS^2 \) and derive a contradiction.

Recall that \( S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \). Define the inclusion map
\[
E_3 : S^2 \to \mathbb{R}^3 \quad \text{by} \quad E_3(x, y, z) = (x, y, z) \in \mathbb{R}^3.
\]
If \( p \in \mathbb{R}^3 \), we can regard \( E_3(p) \) as the outward pointing unit normal to \( S^2 \). Define
\[
E_1 : S^2 \to \mathbb{R}^3 \quad \text{by} \quad E_1(p) = \frac{1}{|X(p)|} X(p).
\]
Finally, use the cross product to define
\[
E_2 : S^2 \to \mathbb{R}^3 \quad \text{by} \quad E_2(p) = E_3(p) \times E_1(p).
\]
For each \( p \in S^2 \), we have a positively oriented orthonormal frame
\[
(E_1(p), E_2(p), E_3(p)) \quad \text{for} \ \mathbb{R}^3.
\]
There is a unique matrix
\[
A(p) = \begin{pmatrix} a_1^1(p) & a_1^2(p) & a_1^3(p) \\ a_2^1(p) & a_2^2(p) & a_2^3(p) \\ a_3^1(p) & a_3^2(p) & a_3^3(p) \end{pmatrix} \in SO(3)
\]
such that

\[(E_1(p) E_2(p) E_3(p)) = (i \ j \ k) \begin{pmatrix} a^1_1(p) & a^1_2(p) & a^1_3(p) \\ a^2_1(p) & a^2_2(p) & a^2_3(p) \\ a^3_1(p) & a^3_2(p) & a^3_3(p) \end{pmatrix}, \]

and we have a smooth map \(A : S^2 \rightarrow SO(3)\) which takes \(p\) to \(A(p)\). We also have a smooth projection \(\pi : SO(3) \rightarrow S^2\), defined by

\[\pi \begin{pmatrix} a^1_1(p) & a^1_2(p) & a^1_3(p) \\ a^2_1(p) & a^2_2(p) & a^2_3(p) \\ a^3_1(p) & a^3_2(p) & a^3_3(p) \end{pmatrix} = \begin{pmatrix} a^1_3(p) \\ a^2_3(p) \\ a^3_3(p) \end{pmatrix}, \]

such that \(\pi \circ A = \text{id}\). Thus the identity map factors through the composition

\[S^2 \xrightarrow{A} SO(3) \xrightarrow{\pi} S^2,\]

and we can apply the contravariant functor \(H^2\) to obtain a factorization of the identity

\[H^2_{dR}(S^2; \mathbb{R}) \xrightarrow{\pi^*} H^2_{dR}(SO(3); \mathbb{R}) \xrightarrow{A^*} H^2_{dR}(S^2; \mathbb{R}).\]

Since \(H^2_{dR}(SO(3); \mathbb{R}) = 0\) and \(H^2_{dR}(S^2; \mathbb{R}) \cong \mathbb{R}\), this yields a contradiction. QED

4.9 The Mayer-Vietoris Sequence*

In order to be able to calculate de Rham cohomology effectively, we need one further property of de Rham cohomology, the exactness of the so-called Mayer-Vietoris sequence. In this section, we will describe the Mayer-Vietoris sequence and show how to use it to calculate the de Rham cohomology of a smooth manifold by dividing it up into simpler pieces.

A sequence of vector spaces and linear maps,

\[\cdots \rightarrow V_{i-1} \xrightarrow{T_{i-1}} V_i \xrightarrow{T_i} V_{i+1} \rightarrow \cdots,\]

extending infinitely far in each direction, is said to be exact if

\[\text{Im}(T_{i-1}) = \text{Ker}(T_i)\]

for each \(i\). In particular, the sequence

\[0 \rightarrow V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \rightarrow 0,\]

is a short exact sequence if

1. \(T_1\) is injective,
2. $T_2$ is surjective, and
3. $\text{Im}(T_1) = \text{Ker}(T_2)$.

Suppose that $U$ and $V$ are open subsets of a smooth manifold $M$ such that $M = U \cup V$. We then have a diagram of inclusion maps:

$$
\begin{array}{c}
U \cap V \xrightarrow{j_U} U \\
\downarrow j_V \downarrow i_U \\
V \xrightarrow{i_V} M
\end{array}
$$

Using this diagram, we can construct a sequence of vector spaces and linear maps

$$
0 \to \Omega^k(M) \xrightarrow{i^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j^*} \Omega^k(U \cap V) \to 0,
$$

(4.22)

where

$$
i^*(\omega) = (i_U^*(\omega), i_V^*(\omega)), \quad j^*(\phi_U, \phi_V) = j_U^*(\phi_U) - j_V^*(\phi_V).
$$

Since $i^*$ and $j^*$ commute with $d$, (4.22) yields a sequence of cochain complexes

$$
0 \to \Omega^*(M) \xrightarrow{i^*} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j^*} \Omega^*(U \cap V) \to 0.
$$

(4.23)

**Lemma 4.9.1.** The sequence (4.23) is exact; in other words for each $k$, the sequence (4.22) of vector spaces and linear maps is exact.

One easily checks that $i^*$ is injective and $j^* \circ i^* = 0$. If $j^*(\phi_U, \phi_V) = 0$, then $j_U^*(\phi_U) = j_V^*(\phi_V)$ so $j_U^*(\phi_U)$ and $j_V^*(\phi_V)$ fit together to form a smooth form on $M$ such that $i^*(\omega) = (\phi_U, \phi_V)$.

The only difficult step in the proof is to show that $j^*$ is surjective. To do this, we choose a partition of unity $\{\psi_U, \psi_V\}$ subordinate to the open cover $\{U, V\}$ of $M$. If $\theta \in \Omega^k(U \cap V)$, we define $\phi_U \in \Omega^k(U)$ by

$$
\phi_U(p) = \begin{cases} 
\psi_V(p)\theta(p), & \text{for } p \in U \cap V, \\
0, & \text{for } p \in U - (U \cap V), 
\end{cases}
$$

and $\phi_V \in \Omega^k(V)$ by

$$
\phi_V(p) = \begin{cases} 
-\psi_U(p)\theta(p), & \text{for } p \in U \cap V, \\
0, & \text{for } p \in V - (U \cap V).
\end{cases}
$$

then

$$
j^*(\phi_U, \phi_V) = \phi_U|(U \cap V) - \phi_V|(U \cap V) = \psi_V\theta + \psi_U\theta = \theta,
$$

so $j^*$ is surjective and the lemma is proven.
The lemma yields a large commutative diagram in which the rows are exact:

\[
\begin{array}{cccc}
  d & d & d \\
0 \rightarrow \Omega^k(M) & \rightarrow \Omega^k(U) \oplus \Omega^k(V) & \rightarrow \Omega^k(U \cap V) \rightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 \rightarrow \Omega^{k+1}(M) & \rightarrow \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) & \rightarrow \Omega^{k+1}(U \cap V) \rightarrow 0 \\
\end{array}
\]

(4.24)

Note that \( i^* \) and \( j^* \) induce homomorphisms on cohomology

\[
i^*: H^k_{dR}(M; \mathbb{R}) \rightarrow H^k_{dR}(U; \mathbb{R}) \oplus H^k_{dR}(V; \mathbb{R}),
\]

\[
j^*: H^{k+1}_{dR}(U; \mathbb{R}) \oplus H^{k+1}_{dR}(V; \mathbb{R}) \rightarrow H^{k+1}_{dR}(U \cap V; \mathbb{R}).
\]

The commuting diagram (4.24) allows us to construct a “connecting homomorphism”

\[
\Delta: H^k_{dR}(U \cap V; \mathbb{R}) \rightarrow H^{k+1}_{dR}(M; \mathbb{R})
\]

as follows: If \( [\theta] \in H^k_{dR}(U \cap V; \mathbb{R}) \), choose a representative \( \theta \in \Omega^k(U \cap V) \).
Since \( j^* \) is surjective, we can choose \( \phi \in \Omega^k(U) \oplus \Omega^k(V) \) so that \( j^*(\phi) = \theta \).
Then \( j^*(d\phi) = dj^*(\phi) = d\theta = 0 \), so there is a unique \( \omega \in \Omega^{k+1}(M) \) such that
\( i^*\omega = d\phi \).
Finally, \( i^*(d\omega) = d(i^*\omega) = d(d\phi) = 0 \), and since \( i^* \) is injective,
\( d\omega = 0 \). Let \([\omega]\) be the de Rham cohomology class of \( \omega \) in \( H^{k+1}_{dR}(M; \mathbb{R}) \) and set
\( \Delta([\theta]) = [\omega] \).
Roughly speaking
\[
\Delta = (i^*)^{-1} \circ d \circ (j^*)^{-1}.
\]

By the technique of “diagram chasing”, one checks that \( \Delta([\theta]) \) is independent of the choice of \( \phi \), or of \( \theta \) representing \( [\theta] \).

We can describe \( \Delta \) explicitly as follows: If \( \theta \) is a representative of \( [\theta] \in H^k_{dR}(U \cap V; \mathbb{R}) \), then \( \Delta([\theta]) \) is represented by the form

\[
d(\psi_U \theta) = d\psi_U \wedge \theta \quad \text{or} \quad -d(\psi_V \theta) = -d\psi_V \wedge \theta,
\]
two expressions for the same \((k + 1)\)-form on \( M \) (since \( \psi_U + \psi_V = 1 \)) which actually has its support in \( U \cap V \).

**Theorem 4.9.2. (Mayer-Vietoris)** The homomorphisms (4.25) and (4.26) fit together to form a long exact sequence

\[
\cdots \rightarrow H^k_{dR}(M; \mathbb{R}) \rightarrow H^k_{dR}(U; \mathbb{R}) \oplus H^k_{dR}(V; \mathbb{R}) \rightarrow H^k_{dR}(U \cap V; \mathbb{R})
\]

\[
\rightarrow H^{k+1}_{dR}(M; \mathbb{R}) \rightarrow H^{k+1}_{dR}(U; \mathbb{R}) \oplus H^{k+1}_{dR}(V; \mathbb{R}) \rightarrow \cdots.
\]

(4.27)
This exact sequence is called the *Mayer-Vietoris sequence*. Together with the Homotopy Lemma, the Mayer-Vietoris sequence is very helpful in computing the de Rham cohomology.

The proof of exactness of the Mayer-Vietoris sequence follows from the so-called “snake lemma” from algebraic topology: A short exact sequence of cochain complexes such as (4.23) gives rise to a long exact sequence in cohomology such as (4.27).

To prove the snake lemma one must establish three assertions:

1. \( \ker(j^*) = \text{im}(i^*) \),
2. \( \ker(\Delta) = \text{im}(j^*) \), and
3. \( \ker(i^*) = \text{im}(\Delta) \).

Each of these assertions is proven by a diagram chase using the diagram (4.24). (These diagram chases are not difficult, but one could argue that they are best done in the privacy of one’s own study. Alternatively, some may prefer to follow the proof of the Snake Lemma in the 1980 movie, “It’s My Turn” starring Jill Clayburgh.)

For example, suppose we want to check the second of these assertions, \( \ker(\Delta) = \text{im}(j^*) \). To see that \( \text{im}(j^*) \subseteq \ker(\Delta) \), we suppose \( \theta \in \text{im}(j^*) \), so \( \theta = j^* \phi \) for some \( \phi \in H^k \oplus H^k(V) \). Then there are representatives \( \theta \) and \( \phi \) such that \( j^* \phi = \theta \). Note that \( d\phi = 0 \). Hence

\[
\Delta([\theta]) = [(i^*)^{-1} \circ d \circ (j^*)^{-1}(\theta)] = 0.
\]

Conversely, suppose that \( [\theta] \in \ker(\Delta) \), so \( (i^*)^{-1} \circ d \circ (j^*)^{-1}(\theta) \) is exact, so if \( \phi \in (j^*)^{-1}(\theta) \), then \( (i^*)^{-1} \circ d\phi = d\omega \), for some \( \omega \in \Omega^k(M) \). Then \( d(\phi - i^*\omega) = 0 \) and \( j^*(\phi - i^*\omega) = \theta \). Thus \( [\theta] \in \text{im}(j^*) \).

The other two assertions are proven by similar arguments. A completely detailed proof would follow exactly the same steps as in the proof of Theorem 2.16 in Hatcher [6].

**Example 1.** We first calculate the cohomology of \( S^1 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \), noting that \( H^0(S^1) \) is isomorphic to the space \( \mathbb{R} \) of constant functions on \( S^1 \). We decompose \( S^1 \) into a union of two open sets

\[
U = S^1 \cap \left\{ y > -\frac{1}{2} \right\}, \quad V = S^1 \cap \left\{ y < \frac{1}{2} \right\},
\]

and note that \( U \) and \( V \) are both diffeomorphic to open intervals, while \( U \cap V \) is diffeomorphic to the union of two open intervals, so the cohomologies of these spaces can be calculated from the Poincaré Lemma. It follows from the Mayer-Vietoris sequence that

\[
0 \to H^0(S^1) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(S^1) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \to \cdots,
\]
which yields
\[ 0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1(S^1) \to 0 \to 0 \to \cdots. \]

It follows that
\[
H^k_{\text{dR}}(S^1; \mathbb{R}) \cong \begin{cases} 
\mathbb{R}, & \text{if } k = 0 \text{ or } k = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

**Example 2.** We next consider \( S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \). Once again \( H^0(S^2) \cong \mathbb{R} \). This time we decompose \( S^2 \) into a union of
\[
U = S^2 \cap \left\{ z > -\frac{1}{2} \right\} \quad \text{and} \quad V = S^2 \cap \left\{ z < \frac{1}{2} \right\}.
\]

In this case, \( U \) and \( V \) are both diffeomorphic to open disks, while \( U \cap V \) is homotopy equivalent to \( S^1 \). It follows from the Mayer-Vietoris sequence that
\[
0 \to H^0(S^2) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \\
\to H^1(S^2) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \\
\to H^2(S^2) \to H^2(U) \oplus H^2(V) \to H^2(U \cap V) \to \cdots,
\]
which yields
\[ 0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1(S^2) \to 0 \to \mathbb{R} \to H^2(S^2) \to 0 \to \cdots. \]

It follows that
\[
H^k_{\text{dR}}(S^2; \mathbb{R}) \cong \begin{cases} 
\mathbb{R}, & \text{if } k = 0 \text{ or } k = 2, \\
0, & \text{otherwise.}
\end{cases}
\]

**Example 3.** By induction, one can calculate the cohomology of \( S^n \):
\[
H^k_{\text{dR}}(S^n; \mathbb{R}) \cong \begin{cases} 
\mathbb{R}, & \text{if } k = 0 \text{ or } k = n, \\
0, & \text{otherwise.}
\end{cases}
\]

The following exercises illustrate how one can use the Poincaré Lemma and the Mayer-Vietoris sequence to calculate the de Rham cohomology of many manifolds. Further examples are given in [3].

**Exercise IXA.** (Do not hand in.) a. Determine the de Rham cohomology of \( \mathbb{R}^2 \) minus a point, which is homotopy equivalent to the disk minus a smaller disk.

b. Use the Mayer-Vietoris sequence to determine the de Rham cohomology of \( \mathbb{R}^2 \) minus \( k \) points, which is homotopy equivalent to the disk minus \( k \) smaller disks, where \( k \in \mathbb{N} \).
c. Use the Mayer-Vietoris sequence to determine
\[ \dim H^k_{dR}(\Sigma_g; \mathbb{R}), \]
where \( \Sigma_g \) is the two-sphere with \( g \) handles, the compact oriented surface of genus \( g \). Hint: Use the fact that \( \Sigma_g \) is orientable and therefore has a volume form which makes the top cohomology nonzero.

**Exercise IXB.** (Do not hand in.)

a. Use the Mayer-Vietoris sequence to determine the de Rham cohomology of \( \mathbb{R}P^2 \).
b. Use the Mayer-Vietoris sequence to determine the de Rham cohomology of \( \mathbb{R}P^3 = SO(3) \), thereby establishing (4.21).

### 4.10 The Hodge star*

In the last few sections, we have described the foundations for algebraic topology from the viewpoint of differential forms on smooth manifolds. Of course, one can regard topology as simply a beautiful edifice in its own right, but it also has numerous applications to the partial differential equations which arise in geometry. This connection between geometry and topology has become increasingly important over the past several decades. Our next goal is to describe one application of de Rham theory, Hodge’s theorem, which states that any de Rham cohomology class contains a unique solution to Laplace’s equation.

The first step in this theory consists of constructing the Hodge star, which generalizes the cross product from \( \mathbb{R}^3 \) to Riemannian manifolds of arbitrary dimension.

To construct the Hodge star, we begin with an oriented Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) of dimension \( n \) without boundary. Recall that for each \( p \in M \), the nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) defines an isomorphism
\[ b : T_p M \to T^*_p M, \quad b(v) = \langle v, \cdot \rangle, \quad \text{with inverse} \quad \sharp : T^*_p M \to T_p M. \]

These isomorphisms allow us to transport the nondegenerate symmetric bilinear form
\[ \langle \cdot, \cdot \rangle : T_p M \times T_p M \to \mathbb{R} \quad \text{to} \quad \langle \cdot, \cdot \rangle : T^*_p M \times T^*_p M \to \mathbb{R}. \]

If \((x^1, \ldots, x^n)\) is a smooth coordinate system on \( U \subseteq M \), and
\[ \langle \cdot, \cdot \rangle|_U = \sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j, \]
then on the cotangent space we have
\[ \langle dx^i|_p, dx^j|_p \rangle = g^{ij}(p), \]
where \((g^{ij}(p))\) is the matrix inverse to \( g_{ij}(p) \).
We can extend $\langle \cdot, \cdot \rangle$ from $T^*_p M$ to the entire exterior algebra $\Lambda^* T^*_p M$. First, we define a multilinear map

$$
\mu : (T^*_p M \times \cdots \times T^*_p M) \times (T^*_p M \times \cdots \times T^*_p M) \to \mathbb{R}
$$

by

$$
\mu((\alpha_1, \ldots, \alpha_k), (\beta_1, \ldots, \beta_k)) = \det \begin{pmatrix}
\langle \alpha_1, \beta_1 \rangle & \cdots & \langle \alpha_1, \beta_k \rangle \\
\langle \alpha_2, \beta_1 \rangle & \cdots & \langle \alpha_2, \beta_k \rangle \\
\vdots & \ddots & \vdots \\
\langle \alpha_k, \beta_1 \rangle & \cdots & \langle \alpha_k, \beta_k \rangle
\end{pmatrix}.
$$

This map is skew-symmetric in each set of $k$ variables, when the other is kept fixed, so it defines a symmetric bilinear form

$$
\langle \cdot, \cdot \rangle : \Lambda^k T^*_p M \times \Lambda^k T^*_p M \to \mathbb{R}
$$

such that

$$
\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle = \det \begin{pmatrix}
\langle \alpha_1, \beta_1 \rangle & \cdots & \langle \alpha_1, \beta_k \rangle \\
\langle \alpha_2, \beta_1 \rangle & \cdots & \langle \alpha_2, \beta_k \rangle \\
\vdots & \ddots & \vdots \\
\langle \alpha_k, \beta_1 \rangle & \cdots & \langle \alpha_k, \beta_k \rangle
\end{pmatrix}. \quad (4.28)
$$

and one can check that this symmetric bilinear form $\langle \cdot, \cdot \rangle$ is positive definite on $\Lambda^k T^*_p M$. One sees this most easily by noting that if $(\theta^1, \ldots, \theta^n)$ is an orthonormal basis for $T^*_p M$, then

$$\{ \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} : i_1 < \cdots < i_k \}$$

is an orthonormal basis for $\Lambda^k T^*_p M$. Hence

$$
\left\langle \sum a_{i_1 \cdots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}, \sum b_{i_1 \cdots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \right\rangle = \sum (a_{i_1 \cdots i_k})^2 \geq 0,
$$

with equality if and only if all the $a_{i_1 \cdots i_k}$'s are zero.

Recall that if $(x^1, \ldots, x^n)$ is a positively oriented coordinate system on $U \subseteq M$, the volume form on $U$ is

$$
\Omega|U = \sqrt{|\det(g_{ij})|} dx^1 \wedge \cdots \wedge dx^n,
$$

and the $\Omega|U$'s fit together to yield a globally defined volume form $\Omega$ on $M$. If $(\theta^1, \ldots, \theta^n)$ is a basis of smooth one-forms on $U$ such that $(\theta^j, \theta^i) = \delta^i_j$, we say it is an orthonormal coframe on $U$. Then $\Omega|U = \pm \theta^1 \wedge \cdots \wedge \theta^n$ and when the plus sign occurs we say that $(\theta^1, \ldots, \theta^n)$ is positively oriented. Using the volume form, we will now define a linear map

$$
* : \Lambda^k T^*_p M \to \Lambda^{n-k} T^*_p M
$$

to be called the Hodge star.
Proposition 4.10.1. For each integer \( k, 0 \leq k \leq n \), there is a unique linear map \( \star : \Lambda^k p^* M \to \Lambda^{n-k} p^* M \) such that whenever \( \alpha \) and \( \beta \) are elements of \( \Lambda^k p^* M \), then
\[
\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \Omega(p),
\]
where \( \Omega(p) \) is the evaluation of the volume form at \( p \).

The idea is to take a fixed positively oriented orthonormal basis \((\theta^1, \ldots, \theta^n)\) for \( T^*_p M \) and derive an explicit formula in terms of this basis.

Given a fixed positively oriented orthonormal basis \((\theta^1, \ldots, \theta^n)\), we claim that (4.29) implies the explicit formula
\[
\star \left( \theta^{\sigma(1)} \wedge \cdots \wedge \theta^{\sigma(k)} \right) = (\text{sgn} \, \sigma) \theta^{\sigma(k+1)} \wedge \cdots \wedge \theta^{\sigma(n)},
\]
whenever \( \sigma \) is a permutation of \( \{1, \ldots, n\} \). Note that formula (4.30) does give a well-defined map on \( \Lambda^k p^* M \) because it is invariant under a transformation which replaces \( \sigma \) by \( \sigma \circ \tau \), whenever \( \tau \) is either a permutation of \( \{1, \ldots, k\} \) or a permutation of \( \{k+1, \ldots, n\} \). Thus to establish (4.30), it suffices to assume without loss of generality that \( \sigma(1) < \cdots < \sigma(k) \) and \( \sigma(k+1) < \cdots < \sigma(n) \).

Under these hypotheses, suppose that
\[
\theta^{j_1} \wedge \cdots \wedge \theta^{j_k} \wedge \star \left( \theta^{\sigma(1)} \wedge \cdots \wedge \theta^{\sigma(k)} \right) = \sum_{j_{k+1} < \cdots < j_n} c_{j_{k+1} \ldots j_n} \theta^{j_{k+1}} \wedge \cdots \wedge \theta^{j_n}.
\]

For a fixed term in this sum, choose \( j_1 < \cdots < j_k \) so that \((j_1, \ldots, j_k, j_{k+1}, \ldots, j_n)\) is a permutation of \((1, \ldots, n)\). Then
\[
\left( \theta^{j_1} \wedge \cdots \wedge \theta^{j_k} \right) \wedge \star \left( \theta^{\sigma(1)} \wedge \cdots \wedge \theta^{\sigma(k)} \right) = c_{j_{k+1} \ldots j_n} \theta^{j_1} \wedge \cdots \wedge \theta^{j_k} \wedge \theta^{j_{k+1}} \wedge \cdots \wedge \theta^{j_n},
\]
all the other terms wedging to zero. On the other hand, it follows from (4.29) that
\[
\left( \theta^{j_1} \wedge \cdots \wedge \theta^{j_k} \right) \wedge \star \left( \theta^{\sigma(1)} \wedge \cdots \wedge \theta^{\sigma(k)} \right) = \begin{cases} 
\theta^1 \wedge \cdots \wedge \theta^n, & \text{if } (j_1, \ldots, j_k) = (\sigma(1), \ldots, \sigma(k)), \\
0, & \text{otherwise}.
\end{cases}
\]

Hence
\[
c_{j_{k+1} \ldots j_n} = \begin{cases}
1, & \text{if } (j_1, \ldots, j_k) = (\sigma(1), \ldots, \sigma(k)), \\
0, & \text{otherwise}.
\end{cases}
\]

It is now easy to verify that
\[
\text{sgn}(\sigma) = c_{\sigma(k+1) \ldots \sigma(n)},
\]
thereby establishing (4.30). This proves uniqueness, because the Hodge star is uniquely determined by its effect on a basis.
To prove existence, one uses (4.30) to define the Hodge star on a fixed orthonormal basis, and checks that it satisfies (4.29). This is an easy verification which we leave to the diligent reader.

Note that it follows from the proof that if \((\theta^1, \ldots, \theta^n)\) is any positively oriented orthonormal basis for \(T^*_p M\), then (4.30) holds for that basis. This fact is extremely useful in calculating the Hodge star.

**Remark:** One can give a very useful geometric interpretation of the Hodge star. Suppose that \(W\) is a \(k\)-dimensional subspace of \(T^*_p M\) with orthonormal basis \((\theta^1, \ldots, \theta^k)\). Complete \((\theta^1, \ldots, \theta^k)\) to a positively oriented orthonormal basis \((\theta^1, \ldots, \theta^n)\) for \(T^*_p M\). Then \((\theta^{k+1}, \ldots, \theta^n)\) is a positively oriented orthonormal basis for the oriented orthogonal complement \(W^\perp\) to \(W\) in \(T^*_p M\).

Of course, the Hodge star extends immediately to a linear map
\[
* : \Omega^k(M) \to \Omega^{n-k}(M)
\]
which is also called the Hodge star. It is easy to verify that
\[
*(*)\alpha = (-1)^{k(n-k)} \alpha, \quad \text{for } \alpha \in \Omega^k(M). \tag{4.31}
\]

**Example 1.** We first consider \(\mathbb{R}^2\) with its usual Euclidean inner product and the Euclidean coordinates \((x, y)\). In this case, \(\Theta = *1 = dx \wedge dy\), and

\[
*(dx) = dy, \quad *(dy) = -dx, \quad \text{so} \quad *(Pdx + Qdy) = -Qdx + Pdy.
\]

We can think of the Hodge star on \(\mathbb{R}^2\) as a counterclockwise rotation through 90 degrees. More generally, if \((M, \langle \cdot, \cdot \rangle)\) is an oriented two-dimensional Riemannian manifold, we can define a **counterclockwise rotation through 90 degrees**
\[
J : \Omega^1(M) \to \Omega^1(M) \quad \text{by} \quad J(\omega) = *(\omega).
\]

**Example 2.** We next consider \(\mathbb{R}^3\) with its usual Euclidean inner product and the Euclidean coordinates \((x, y, z)\). In this case, \(\Theta = *1 = dx \wedge dy \wedge dz\). Moreover,

\[
*(dx) = dy \wedge dz, \quad *(dy) = dz \wedge dx, \quad *(dz) = dx \wedge dy,
\]

\[
*(dy \wedge dz) = dx, \quad *(dz \wedge dx) = dy, \quad *(dx \wedge dy) = dz.
\]

Note that if \(\alpha\) and \(\beta\) are elements of \(\Omega^1(\mathbb{E}^3)\), then so is \(* (\alpha \wedge \beta)\), and one can check that its components are the same as those of the cross product \(\alpha \times \beta\). More generally, if \((M, \langle \cdot, \cdot \rangle)\) is an oriented three-dimensional Riemannian manifold we can define a **cross product**
\[
\times : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M) \quad \text{by} \quad \alpha \times \beta = * (\alpha \wedge \beta).
\]
4.11 Lorentz manifolds; Maxwell’s equations*

(Note: This section provides an application to physics which can be skipped without loss of continuity.) The notion of Riemannian manifold has a generalization which is extremely useful in Einstein’s theory of general relativity [14] and the Hodge star can be extended to this context. This provides a simplification of the equations of electricity and magnetism and gives them a de Rham cohomology interpretation—one of the reasons that de Rham cohomology is important to physics.

**Definition.** Let $M$ be a smooth manifold. A **pseudo-Riemannian metric** on $M$ is a function which assigns to each $p \in M$ a nondegenerate symmetric bilinear map

$$\langle \cdot, \cdot \rangle_p : T_p M \times T_p M \to \mathbb{R}$$

which which varies smoothly with $p \in M$. (Positive definite implies nondegenerate, so any Riemannian metric is also a pseudo-Riemannian metric.) As before, varying smoothly with $p \in M$ means that if $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ is a smooth coordinate system on $M$, then for $p \in U$,

$$\langle \cdot, \cdot \rangle_p = \sum_{i,j=1}^{n} g_{ij}(p) dx^i|_p \otimes dx^j|_p,$$

where the functions $g_{ij} : U \to \mathbb{R}$ are smooth. The conditions that $\langle \cdot, \cdot \rangle_p$ be symmetric and nondegenerate are expressed by saying that $(g_{ij})$ is a symmetric matrix and has nonzero determinant.

It follows from linear algebra that for any choice of $p \in M$, local coordinates $(x^1, \ldots, x^n)$ can be chosen so that

$$(g_{ij}(p)) = \begin{pmatrix} -I_{p \times p} & 0 \\ 0 & I_{q \times q} \end{pmatrix},$$

where $I_{p \times p}$ and $I_{q \times q}$ are $p \times p$ and $q \times q$ identity matrices with $p + q = n$. The pair $(p, q)$ is called the signature of the pseudo-Riemannian metric. A pseudo-Riemannian metric of signature $(0, n)$ is just a Riemannian metric. A pseudo-Riemannian metric of signature $(1, n-1)$ is called a Lorentz metric.

A **pseudo-Riemannian manifold** is a pair $(M, \langle \cdot, \cdot \rangle)$ where $M$ is a smooth manifold and $\langle \cdot, \cdot \rangle$ is a pseudo-Riemannian metric on $M$. Similarly, a **Lorentz manifold** is a pair $(M, \langle \cdot, \cdot \rangle)$ where $M$ is a smooth manifold and $\langle \cdot, \cdot \rangle$ is a Lorentz metric on $M$.

**Example.** Let $\mathbb{R}^{n+1}$ be given coordinates $(t, x^1, \ldots, x^n)$, with $t$ being regarded as time and $(x^1, \ldots, x^n)$ being regarded as Euclidean coordinates in space, and consider the Lorentz metric

$$\langle \cdot, \cdot \rangle = -c^2 dt \otimes dt + \sum_{i=1}^{n} dx^i \otimes dx^i,$$
where the constant \( c \) is regarded as the speed of light. When endowed with this metric, \( \mathbb{R}^{n+1} \) is called Minkowski space-time and is denoted by \( \mathbb{L}^{n+1} \). Four-dimensional Minkowski space-time is the arena for special relativity.

The arena for general relativity is a more general four-dimensional Lorentz manifold \( (M, \langle \cdot, \cdot \rangle) \), also called space-time. In the case of general relativity, the components \( g_{ij} \) of the metric are regarded as potentials for the gravitational forces.

In either case, points of space-time can be thought of as events that happen at a given place in space and at a given time. The trajectory of a moving particle can be regarded as a curve of events, called its world line.

If \( p \) is an event in a Lorentz manifold \( (M, \langle \cdot, \cdot \rangle) \), the tangent space \( T_p M \) inherits a Lorentz inner product \( \langle \cdot, \cdot \rangle_p : T_p M \times T_p M \longrightarrow \mathbb{R} \).

We say that an element \( v \in T_p M \) is

1. **timelike** if \( \langle v, v \rangle < 0 \),
2. **spacelike** if \( \langle v, v \rangle > 0 \), and
3. **lightlike** if \( \langle v, v \rangle = 0 \).

A parametrized curve \( \gamma : [a, b] \rightarrow M \) into a Lorentz manifold \( (M, \langle \cdot, \cdot \rangle) \) is said to be timelike if \( \gamma'(u) \) is timelike for all \( u \in [a, b] \). If a parametrized curve \( \gamma : [a, b] \rightarrow M \) represents the world line of a massive object, it is timelike and the integral

\[
L(\gamma) = \frac{1}{c} \int_a^b \sqrt{-\langle \gamma'(u) \gamma'(u) \rangle} du \quad (4.32)
\]

is the elapsed time measured by a clock moving along the world line \( \gamma \). We call \( L(\gamma) \) the elapsed proper time along \( \gamma \).

**The Twin Paradox.** The fact that elapsed time is measured by the integral (4.32) has counterintuitive consequences. Suppose that \( \gamma : [a, b] \rightarrow \mathbb{L}^4 \) is a timelike curve in four-dimensional Minkowski space-time, parametrized so that

\[
\gamma(t) = (t, x^1(t), x^2(t), x^3(t)).
\]

Then

\[
\gamma'(t) = \frac{\partial}{\partial t} + \sum_{i=1}^3 \frac{dx^i}{dt} \frac{\partial}{\partial x^i}, \quad \text{so} \quad \langle \gamma'(t), \gamma'(t) \rangle = -c^2 + \sum_{i=1}^3 \left( \frac{dx^i}{dt} \right)^2,
\]

and hence

\[
L(\gamma) = \int_a^b \frac{1}{c} \sqrt{c^2 - \sum_{i=1}^3 \left( \frac{dx^i}{dt} \right)^2} dt = \int_a^b \sqrt{1 - \frac{1}{c^2} \sum_{i=1}^3 \left( \frac{dx^i}{dt} \right)^2} dt. \quad (4.33)
\]
Thus if a clock is at rest with respect to the coordinates, that is \( dx^i/dt \equiv 0 \), it will measure the time interval \( b - a \), while if it is in motion it will measure a somewhat shorter time interval. This failure of clocks to synchronize is what is called the twin paradox. An example of this is provided by the differential aging of characters in the recent Hollywood film, “Interstellar.”

The Hodge star can be extended to arbitrary pseudo-Riemannian manifolds. To do this one first defines a nondegenerate symmetric bilinear form

\[
\langle \cdot, \cdot \rangle : \Lambda^k T_p^* M \times \Lambda^k T_p^* M \rightarrow \mathbb{R}
\]

by (4.28) and says that a basis \((\theta^1, \ldots, \theta^n)\) for \( T_p^* M \) is orthonormal if

\[
(\langle \theta^i, \theta^j \rangle) = \begin{pmatrix} -I_{p \times q} & 0 \\ 0 & I_{q \times q} \end{pmatrix},
\]

where \( p + q = n \). There are two possible signs for the volume form

\[
\Omega|U = \pm \theta^1 \wedge \cdots \wedge \theta^n.
\]

This sign depends upon convention which might differ from author to author. Once the sign is chosen one defines the Hodge star \( \star : \Lambda^k T_p^* M \rightarrow \Lambda^{n-k} T_p^* M \) by

\[
\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \Omega(p),
\]

just as in the Riemannian case.

**Example.** Let us consider the Hodge star in Minkowski space-time \( L^4 \) with its standard coordinates \((t, x, y, z)\), taking the speed of light to be \( c = 1 \) so that the Lorentz metric is

\[
\langle \cdot, \cdot \rangle = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz. 
\]

(4.34)

In this case, the volume form is

\[
\Omega = \star 1 = dt \wedge dx \wedge dy \wedge dz.
\]

Clearly, \( \star (dt \wedge dx) = \pm dy \wedge dz. \) To determine the sign, we note that

\[
\langle dt \wedge dx, dt \wedge dx \rangle = -1, \text{ so } (dt \wedge dx)(\star dt \wedge dx) = -\Omega.
\]

It follows that

\[
\star (dt \wedge dx) = -dy \wedge dz, \quad \star (dt \wedge dy) = -dz \wedge dy, \quad \star (dt \wedge dz) = -dx \wedge dy.
\]

By similar arguments, one verifies that

\[
\star (dy \wedge dz) = dt \wedge dx, \quad \star (dz \wedge dx) = dt \wedge dy, \quad \star (dx \wedge dy) = dt \wedge dz.
\]

This Hodge star is invariant under orientation-preserving Lorentz transformation, those orientation-preserving linear transformations of \( L^4 \) which leave invariant the flat Lorentz metric (4.34).
Maxwell’s equations. Using the Hodge star on Minkowski space-time with the standard Lorentz metric, we can give a particularly elegant formulation of Maxwell’s equations from electricity and magnetism, equations formulated by James Clerk Maxwell in 1873. This formulation has the advantage that it extends to the curved space-times of general relativity.

In Maxwell’s theory of electricity and magnetism, one imagines that one is given the charge density $\rho(t, x, y, z)$ and the current density

$$J(t, x, y, z) = J_x \frac{\partial}{\partial x} + J_y \frac{\partial}{\partial y} + J_z \frac{\partial}{\partial z}.$$ 

The charge and current density should determine the electric and magnetic fields

$$E(t, x, y, z) = E_x \frac{\partial}{\partial x} + E_y \frac{\partial}{\partial y} + E_z \frac{\partial}{\partial z} = \text{(electric field)}$$

and

$$B(t, x, y, z) = B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} = \text{(magnetic field)}$$

by means of Maxwell’s equations, which are expressed in terms of the divergence and curl operations studied in second year calculus as

$$\nabla \cdot B = 0, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad (4.35)$$

$$\nabla \cdot E = 4\pi \rho, \quad \nabla \times B - \frac{\partial E}{\partial t} = 4\pi J. \quad (4.36)$$

To express these equations in space-time formalism, it is convenient to replace the electric and magnetic fields by a single covariant tensor field of rank two, the so called Faraday tensor:

$$\mathcal{F} = -E_x dt \wedge dx - E_y dt \wedge dy - E_z dt \wedge dz$$

$$\quad + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$ 

Then the Hodge star interchanges the $E$ and the $B$ fields:

$$\star \mathcal{F} = B_x dt \wedge dx + B_y dt \wedge dy + B_z dt \wedge dz$$

$$\quad + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy.$$ 

ExerciseXA. (Do not hand in.) a. Show that in Minkowski space-time $\mathbb{L}^4$,

$$\star \star = (-1)^{k+1} : \Omega^k(\mathbb{L}^4) \to \Omega^k(\mathbb{L}^4).$$

b. Determine $\star dt, \star dx, \star dy$ and $\star dz$.

c. Show that Maxwell’s equations can be expressed in terms of the Faraday tensor as

$$d\mathcal{F} = 0, \quad d(\star \mathcal{F}) = \star(4\pi \mathcal{J}),$$

where

$$\mathcal{J} = -\rho dt + J_x dx + J_y dy + J_z dz. \quad (4.37)$$

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The formulation (4.37) of Maxwell’s equations is described in more detail in Chapter 4 of Misner, Thorne and Wheeler’s excellent book, *Gravitation* [14]. When expressed in terms of the Hodge star, it is apparent that Maxwell’s equations are invariant under the action of the Lorentz group, the linear transformations from $L^4$ to itself which preserve the Lorentz metric of Minkowski space-time, since the Hodge star is completely defined by the Lorentz metric. This invariance was one of the clues that led Einstein to the discovery of special relativity—Maxwell’s equations were not invariant under the same group of transformations as Newtonian mechanics. Moreover, the formulation of Maxwell’s equations in terms of the Faraday tensor can be extended immediately to the curved space-times of general relativity. Maxwell’s equations thus provide a motivation for the study of the operator $\star d \star$ on differential forms in higher dimensions.

One approach to solving Maxwell’s equations on $L^4$ is to write $F = dA$, where $A$ is a one-form called the vector potential. Appropriately chosen, the vector potential will solve an equation similar to the wave equation, for which an elegant theory has been developed. This approach runs into a snag in a general space-time because the de Rham cohomology class $[F]$ might be nonzero, in which case no vector potential can be found. Resolving this question leads to the theory of connections in vector bundles over space-time.

### 4.12 The Hodge Laplacian*

In addition to the exterior derivative, an $n$-dimensional Riemannian manifolds has a codifferential $\delta$ which goes in the opposite direction,

$$
\delta = (-1)^{nk+1} \star d \star : \Omega^{k+1}(M) \rightarrow \Omega^k(M).
$$

The only thing that is difficult to remember about the codifferential is the sign. For now, note that if $M$ is even-dimensional, $\delta = - \star d \star$. No matter what the dimension of $M$, it follows from (4.31) that $\delta \circ \delta = 0$.

To explain where the sign comes from, we suppose that $(M, \langle \cdot, \cdot \rangle)$ is a compact oriented Riemannian manifold, possibly with boundary $\partial M$. We can define a positive definite $L^2$ inner product

$$
(\cdot, \cdot) : \Omega^k(M) \times \Omega^k(M) \rightarrow \mathbb{R}
$$

by setting

$$
(\phi, \psi) = \int_M \phi \wedge \star \psi = \int_M \langle \phi, \psi \rangle \Theta.
$$

This inner product makes $\Omega^k(M)$ into a pre-Hilbert space. The funny sign is introduced to make the following proposition valid:

**Proposition 4.12.1.** If $\phi \in \Omega^k(M)$ and $\psi \in \Omega^{k+1}(M)$, then

$$
(d\phi, \psi) - (\phi, \delta \psi) = \int_{\partial M} \phi \wedge \star \psi,
$$

(4.38)
where $\delta = (-1)^{nk+1} \ast d\ast$.

The proposition is a consequence of Stokes’ Theorem:

$$\int_{\partial M} \phi \wedge \ast \psi = \int_M d(\phi \wedge \ast \psi) = \int_M \phi \wedge (d \ast \psi) = \int_M \phi \wedge (-1)^k (-1)^{(n-k)k} \ast d \ast \psi$$

$$= \int_M \phi \wedge \ast \psi - \int_M \ast(\phi \wedge (\ast d \ast \psi))$$

Suppose $p \in \partial M$ and that $\nu$ is a unit-length element of $T^*_pM$ which points out of $\partial M$. The volume forms $\Omega_M$ and $\Omega_{\partial M}$ are then related by the formula

$$\Omega_M = \nu \wedge \Omega_{\partial M}.$$

If $\phi \in \Omega^k(M)$ and $\psi \in \Omega^{k+1}(M)$, then

$$\nu \wedge \phi \wedge \ast \psi = \langle \nu \wedge \phi, \psi \rangle \Omega_M \Rightarrow \phi \wedge \ast \psi = \langle \nu \wedge \phi, \psi \rangle \Omega_{\partial M},$$

so we can rewrite (4.38) as

$$(d\phi, \psi) - (\phi, \delta \psi) = \int_{\partial M} \langle \nu \wedge \phi, \psi \rangle \Omega_{\partial M}. \quad (4.39)$$

If we also use the Riemannian metric to identify $\nu$ with a unit length tangent vector, it follows from the identity $\langle \nu \wedge \phi, \psi \rangle = \langle \phi, \iota_\nu \psi \rangle$, where $\iota_\nu$ is the interior product discussed in Exercise VIIIA, that we can write this formula as

$$(d\phi, \psi) - (\phi, \delta \psi) = \int_{\partial M} \langle \phi, \iota_\nu \psi \rangle \Omega_{\partial M}. \quad (4.40)$$

Note that if $\partial M = \emptyset$, equation (4.38) becomes

$$(d\phi, \psi) = (\phi, \delta \psi). \quad (4.41)$$

The sign in the definition of $\delta$ was chosen to make this identity hold. Because of the identity, the codifferential $\delta$ is also called the formal adjoint to $d$.

Given an oriented Riemannian manifold, possibly with boundary, we now have two first order differential operators $d$ and $\delta$ which satisfy the identities $d^2 = 0$ and $\delta^2 = 0$. It follows that

$$\Delta = -(d + \delta)^2 = -d\delta - \delta d : \Omega^k(M) \rightarrow \Omega^k(M).$$

**Definition.** The Hodge Laplacian is the second order differential operator

$$\Delta : \Omega^k(M) \rightarrow \Omega^k(M) \text{ defined by } \Delta = -(d\delta + \delta d).$$

We say that an element $\phi \in \Omega^k(M)$ is harmonic if $\Delta \phi = 0$. 

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**Dangerous curve.** Most geometry books use the opposite sign in the definition of the Hodge Laplacian. We have chosen the sign so that the Hodge Laplacian agrees with the Laplace operator used by engineers and physicists when $M$ is Euclidean space.

Suppose that $(M, \langle \cdot, \cdot \rangle)$ is a compact Riemannian manifold without boundary. It then follows from the identity (4.41) that

$$( -\Delta \phi, \psi ) = ( (d\delta + \delta d)\phi, \psi ) = ( \delta \phi, \delta \psi ) + ( d\phi, d\psi ) = \cdots = ( \phi, -\Delta \psi ), \quad (4.42)$$

for all $\phi, \psi \in \Omega^k(M)$. Equation (4.42) states that the Laplace operator $\Delta$ is *formally self-adjoint*. Moreover, $\Delta \phi = 0$ if and only if $\phi$ satisfies the *weak form* of Laplace’s equation on $k$-forms:

$$( \delta \phi, \delta \psi ) + ( d\phi, d\psi ) = 0, \quad \text{for all } \psi \in \Omega^k(M). \quad (4.43)$$

We can take $\psi = \phi$, so that

$$\Delta \phi = 0 \quad \Rightarrow \quad (\delta \phi, \delta \phi) + (d\phi, d\phi) = 0.$$  

Since the inner product $\langle \cdot, \cdot \rangle$ is positive definite, it follows that $d\phi = 0 = \delta \phi$, and we conclude:

**Proposition 4.12.2.** If $(M, \langle \cdot, \cdot \rangle)$ is a compact oriented Riemannian manifold without boundary, harmonic $k$-forms on $M$ are exactly those forms which are both closed and coclosed:

$$\Delta \phi = 0 \iff d\phi = 0 = \delta \phi. \quad (4.44)$$

**The Laplace operator on functions.** To gain some intuition, we focus on the simplest case, the Laplace operator on functions. In this case,

$$\Delta(f) = -\delta d(f), \quad \text{since} \quad \delta(f) = 0$$

and $\delta = -\ast d\ast$.

We imagine that $(x^1, \ldots, x^n)$ is a smooth positively oriented coordinate system on $M$, so that

$$\Omega = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n.$$  

Clearly,

$$\ast(dx^i) = \sum_{j=1}^{n} (-1)^{j-1} h^{ij} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n,$$

for certain functions $h^{ij}$. Hence

$$dx^i \wedge \ast dx^j = \cdots = h^{ij} dx^1 \wedge \cdots \wedge dx^n.$$
On the other hand, it follows from (4.29) that
\[ \langle dx^i, dx^j \rangle = g^{ij} \sqrt{g} dx^1 \wedge \cdots \wedge dx^n. \]
It follows that \( h^{jk} = g^{jk} \sqrt{g} \), and hence
\[ *(dx^i) = \sum_{j=1}^{n} (-1)^{j-1} g^{ij} \sqrt{g} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n. \]

It follows that
\[ *\left(df\right) = \left( \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \right) \]
\[ = \sum_{i,j=1}^{n} (-1)^{j-1} g^{ij} \sqrt{g} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n, \]
and
\[ d \star d(f) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x^j} \left( g^{ij} \sqrt{g} \frac{\partial f}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n. \]

Thus we finally conclude that
\[ \Delta(f) = \star d \star df = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^j} \left( g^{ij} \sqrt{g} \frac{\partial f}{\partial x^i} \right). \]  
(4.45)

This formula for the Laplace operator is incredibly useful. Consider, for example, the flow of heat on a smooth surface \( M \subseteq \mathbb{E}^3 \). From the elementary theory of PDE’s we expect the temperature on a smooth homogeneous surface to be described by a function \( u : M \times [0, \infty) \to \mathbb{R} \) which satisfies an initial value problem
\[ \frac{\partial u}{\partial t} = c^2 \Delta u, \quad u(p, 0) = h(p), \]
where \( h : M \to \mathbb{R} \) is the initial temperature distribution. However, in order to make sense of this equation, we need to define a Laplace operator acting on scalar functions on a smooth surface. The Laplace operator to use is the one given by (4.45).

**Exercise XB. (Do not hand in.)** Suppose that \( M = S^2 \), the standard unit two-sphere in \( \mathbb{E}^3 \), with Riemannian metric expressed in spherical coordinates as
\[ \langle \cdot, \cdot \rangle = (\sin^2 \phi) d\theta \otimes d\theta + d\phi \otimes d\phi. \]
Determine the Hodge Laplacian on functions in this case.

**Remark.** We could use this expression for the Laplacian to solve the initial-value problem for the flow of heat over the unit sphere. The technique of separation of variables and Legendre polynomials gives a very explicit representation of the solution.
Suppose now that \((M, \langle \cdot, \cdot \rangle)\) is a compact oriented Riemannian manifold without boundary, and let
\[
\mathcal{H}^k(M) = \{ \text{harmonic } k\text{-forms on } M \} = \{ \phi \in \Omega^k(M) : \Delta \phi = 0 \}.
\]

An amazing result relates de Rham cohomology to solutions to the Laplace equation:

**Theorem 4.12.3. (Hodge Theorem)** Every de Rham cohomology class in a smooth compact Riemannian manifold without boundary has a unique harmonic representative; thus
\[
H^k_{dR}(M; \mathbb{R}) \cong \mathcal{H}^k(M).
\]

This gives an important relationship between topology and solutions to linear elliptic systems of partial differential equations on smooth manifolds. We will not prove this theorem in the course, but refer instead to the last chapter of the text by Warner [18].

This theorem has important topological consequences. Thus for example, one easily checks that \(\ast \Delta = \Delta \ast\), so if \(\phi\) is a harmonic \(k\)-form, so is \(\ast \phi\). Thus we obtain:

**Theorem 4.12.4. (Poincaré Duality)** If \(M\) is a compact oriented smooth manifold without boundary of dimension \(n\),
\[
H^k_{dR}(M; \mathbb{R}) \cong H^{n-k}_{dR}(M; \mathbb{R}).
\]

We already know that if \(M\) is a compact oriented Riemannian manifold, the volume form represents a nontrivial element of \(H^n_{dR}(M; \mathbb{R})\). Poincaré duality enables us to make a finer statement:

**Corollary 4.12.5.** If \(M\) is a compact connected oriented smooth manifold without boundary of dimension \(n\),
\[
H^n_{dR}(M; \mathbb{R}) \cong \mathbb{R}.
\]

Clearly, Hodge theory and Poincaré duality simplify the calculation of de Rham cohomology in many cases.

**Example.** Let us suppose that
\[
M = T^n = S^1 \times \cdots \times S^1,
\]
with the flat metric defined by requiring that the covering
\[
\pi : \mathbb{R}^n \to T^n, \quad \pi(x^1, \ldots, x^n) = (e^{2\pi i x^1}, \ldots, e^{2\pi i x^n})
\]
be a local isometry. We define one-forms \((\theta^1, \ldots, \theta^n)\) on \(T^n\) by \(\pi^* \theta^i = dx^i\). Then \((\theta^1, \ldots, \theta^n)\) form a positively oriented orthonormal basis for the one-forms on \(M\) and it follows from (4.30) that

\[
d(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = 0, \quad \delta(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = 0.
\]

If \(\omega \in \Omega^k(T^n)\), then

\[
\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \ldots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k},
\]

for some smooth real-valued functions \(f_{i_1 \ldots i_k}\) on \(T^n\) and if these functions are constant, \(d\omega = 0 = \delta\omega\) and hence \(\omega\) is harmonic. Conversely, a direct calculation shows that

\[
\Delta \omega = \sum_{i_1 < \cdots < i_k} (\Delta f_{i_1 \ldots i_k}) \theta^{i_1} \wedge \cdots \wedge \theta^{i_k},
\]

and hence if \(\omega\) is harmonic, so is each function \(f_{i_1 \ldots i_k}\). But it follows from (4.44) that the only harmonic functions on a compact oriented manifolds are constant on each connected component, so the only harmonic forms on \(T^n\) are

\[
\omega = \sum_{i_1 < \cdots < i_k} c_{i_1 \ldots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k},
\]

where the \(c_{i_1 \ldots i_k}\)'s are constants. In other words,

\[
\{\theta^{i_1} \wedge \cdots \wedge \theta^{i_k} : i_1 < \cdots < i_k\}
\]

is a basis for \(H^k(T^n)\) and hence the dimension of \(H^k_{dR}(T^n; \mathbb{R})\) is \(\binom{n}{k}\). Thus Hodge theory yields a very explicit representation for the cohomology of the torus in terms of harmonic forms.
Bibliography


